CONTRIBUTIONS TO THE THEORY OF SUBNORMAL AND ASCENDANT SUBGROUPS

by

Jennifer Whitehead

A dissertation submitted for the degree of

Doctor of Philosophy

in the University of Warwick

1975

CONTENTS

Acknowledgments		(i)
Abstract		(ii)
Introduction		1
Notation and Terminology		7

PART ONE

Residuals and the Lower Central Series of
Subnormal and Ascendant Subgroups

Chapter 1	The Locally Nilpotent Residual of the Join of two Subnormal Subgroups	13
Chapter 2	The Lower Central Series of the Join of two Subnormal Subgroups	26
Chapter 3	Residuals of the Join of two Ascendant Subgroups	37

PART TWO

Criteria for Subnormality and Ascendancy in Soluble
and Generalized Soluble Groups

Chapter 4	Subnormality and Ascendancy in Soluble and Generalized Soluble Groups	45
Chapter 5	Subnormality in Soluble Groups with Finiteness Conditions	60
Appendix		78
References		82

ACKNOWLEDGEMENTS

I would like to especially thank my supervisor, Dr. Stewart Stonehewer, who has been a constant source of encouragement and ideas, and who has helped me over many difficulties. Thanks are also extended to Dr. Brian Hartley, for his interest and supervision during the first six months of 1974, and whose ideas helped to shape the second part of the thesis.

I am grateful to the Science Research Council for its financial support during the period 1971-1974.

Humble thanks go to Nalsey Tinberg who has so patiently and cheerfully done the difficult job of typing.

Finally, I would like to thank Michael Maller who has offered me invaluable support, friendship and encouragement during the months of its preparation.

Abstract

The thesis is divided into two main parts. We begin with a general introduction and then a section on notation and terminology.

In part one, consisting of the first three chapters, we are concerned with residuals of the join of two subnormal or ascendant subgroups, and the lower central series of the join of two subnormal subgroups. The results generalize those of Wielandt [35].

In chapter one, we prove some elementary results on $\mathfrak{X}$-residuals, for a class of groups $\mathfrak{X}$, and examine, in particular the locally nilpotent residual of a locally finite group which is the join of two subnormal subgroups, and the locally nilpotent residual of the join of two subnormal subgroups satisfying the minimal condition for subnormal subgroups.

In chapter two we prove results on the lower central series of the join, G, of two subnormal subgroups H and K in the cases when (i) G satisfies the minimal condition on normal subgroups and (ii) when H and K satisfy the maximal condition for subnormal subgroups.

In chapter three, residuals of the join, G, of two ascendant subgroups are studied when G is a locally finite group and when H and K satisfy the minimal condition for subnormal subgroups.

In part two, we examine criteria for subnormality and ascendancy in soluble and generalized soluble groups.

In chapter four we extend some of the recent work of Wielandt on criteria for subnormality in finite groups [36], to soluble groups and various classes of generalized soluble groups.

In chapter five we consider conditions imposed on the generators of a finitely generated subgroup H of a soluble group, which imply subnormality subject to certain additional finiteness conditions. We examine first the case of a finite

Abstract

The thesis is divided into two main parts. We begin with a general introduction and then a section on notation and terminology.

In part one, consisting of the first three chapters, we are concerned with residuals of the join of two subnormal or ascendant subgroups, and the lower central series of the join of two subnormal subgroups. The results generalize those of Wielandt [35].

In chapter one, we prove some elementary results on $\mathfrak{X}$-residuals, for a class of groups $\mathfrak{X}$, and examine, in particular the locally nilpotent residual of a locally finite group which is the join of two subnormal subgroups, and the locally nilpotent residual of the join of two subnormal subgroups satisfying the minimal condition for subnormal subgroups.

In chapter two we prove results on the lower central series of the join, G, of two subnormal subgroups H and K in the cases when (i) G satisfies the minimal condition on normal subgroups and (ii) when H and K satisfy the maximal condition for subnormal subgroups.

In chapter three, residuals of the join, G, of two ascendant subgroups are studied when G is a locally finite group and when H and K satisfy the minimal condition for subnormal subgroups.

In part two, we examine criteria for subnormality and ascendancy in soluble and generalized soluble groups.

In chapter four we extend some of the recent work of Wielandt on criteria for subnormality in finite groups [36], to soluble groups and various classes of generalized soluble groups.

In chapter five we consider conditions imposed on the generators of a finitely generated subgroup H of a soluble group, which imply subnormality subject to certain additional finiteness conditions. We examine first the case of a finite

soluble group.

The work in this thesis, apart from the results attributed to others, is to the best of my knowledge, original.

Introduction

Subnormality was first studied in 1939 by Wielandt [34]. He proves several fundamental results, here, for groups with a composition series; in particular that the join of two subnormal subgroups in this class is subnormal, and he shows how his proof can be extended to groups with the maximal condition for subgroups. The join J of two subnormal subgroups H and K need not, however, be subnormal in an arbitrary group, as an example by Zassenhaus shows ([37] Appendix D, Exercise 23). Even when H is an elementary abelian 2-group and K is of order 2, J need not be subnormal as an unpublished example by P. Hall shows. Other such examples are given in [31].

In the early 1960's Robinson [22] examined conditions imposed on a group G which ensure that joins of finitely many subnormal subgroups of G are subnormal. Since the intersection of finitely many subnormal subgroups is a subnormal subgroup, the groups for which the join property holds are just the class of groups whose set of subnormal subgroups is a lattice (with respect to the operations set intersection and group theoretical join). Denoting this class of groups by $\mathfrak{L}$ in [22] Robinson shows that if G is a group whose derived subgroup G' satisfies the maximal condition for subnormal subgroups, then G belongs to $\mathfrak{L}$, and that an extension of a group in the class $\mathfrak{L}$ by a group which satisfies the maximal condition for subgroups belongs to the class $\mathfrak{L}$. He also examines conditions imposed on a pair of subnormal subgroups which will make their join subnormal.

A subnormal coalition class is a class of groups, $\mathfrak{X}$, for which a pair of subnormal $\mathfrak{X}$-subgroups generates a subnormal $\mathfrak{X}$- subgroup. These have been studied in the literature and include the class of finite, finite π-groups and polycyclic groups (Wielandt [34]), groups with the maximal condition for subgroups (Baer [3]), finitely generated nilpotent groups (Baer [2]), groups with maximal or minimal condition on subnormal subgroups (Robinson [23]; Roseblade [27] [29]), minimax groups (Roseblade unpublished), any subjunctive class of finitely generated groups [31], and more recently the class of groups with finite rank [5].

It is well-known that the join of two nilpotent subnormal subgroups need not be nilpotent, and in fact the Zassenhaus example [37] shows that this is so. Even when G is generated by two subnormal abelian subgroups of exponent 2, their join need not be nilpotent (see for example [31]). The fact that in all such examples as these, the join, although not nilpotent, was soluble, led to the conjecture that the join of two soluble subnormal subgroups was soluble. This had been proved by Wielandt in [34] for groups with a composition series, and in 1970 Stonehewer [32] showed it was true in general.

The nilpotency of the join J, of two nilpotent subnormal subgroups H and K, under addition hypotheses, are found in the literature, however. These include when H and K are finitely generated (Baer [2]), H and K satisfy the minimal condition for subnormal subgroups (Robinson [23] and Roseblade [27]), H and K satisfy the maximal condition for subnormal subgroups (Roseblade [29]) and H and K have finite rank [5].

Denoting the class of nilpotent groups by $\mathfrak{N}$ and the limit of the lower central series by $G^{\mathfrak{N}}$, in 1957 Wielandt ([35] Satz 1.5) showed more generally that if H and K are subnormal subgroups of a group G with a composition series, then

$$H^{\mathfrak{N}}K = KH^{\mathfrak{N}} \quad \text{and} \quad \langle H, K\rangle^{\mathfrak{N}} = H^{\mathfrak{N}}K^{\mathfrak{N}} \quad (1)$$

Again, denoting the class of soluble groups by $\mathfrak{S}$, and the limit of the derived series by $G^{\mathfrak{S}}$, Wielandt has shown ([34] Satz 2.3 and [35] Satz 2.4)

$$H^{\mathfrak{S}}K = KH^{\mathfrak{S}} \quad \text{and} \quad \langle H, K\rangle^{\mathfrak{S}} = H^{\mathfrak{S}}K^{\mathfrak{S}} \quad (2)$$

In both these results, the condition that G has a composition series cannot be removed, for let A be any group. Then by

[9] lemma 13, A can be embedded in a normal product G of two free subgroups H and K. But by [13] $H^{\mathfrak{N}} = H^{\mathfrak{S}} = K^{\mathfrak{N}} = K^{\mathfrak{S}} = 1$ and so if $G^{\mathfrak{S}} = G^{\mathfrak{N}} = 1$, this would imply $A^{\mathfrak{N}} = A^{\mathfrak{S}} = 1$ for any group A.

However, in [32], Stonehewer shows that (2) holds provided that $H/H^{\mathfrak{S}}$ and $K/K^{\mathfrak{S}}$ are soluble. In [30] Roseblade has generalized this still further, and shown that denoting the n-th term of the derived series of a group G by $G^{(n)}$, then if J is the join of two subnormal subgroups H and K, then given any two positive integers r_1 and r_2 there exists an integer r such that

$$J^{(r)} \leq H^{(r_1)} K^{(r_2)} \qquad (3)$$

More recently, Stonehewer has shown in [33] that if H and K have finite rank, then (1) holds provided that $H/H^{\mathfrak{N}} \in \mathfrak{N}$ and $K/K^{\mathfrak{N}} \in \mathfrak{N}$.

In chapter 1 of the thesis we shall prove a result analogous to (1) for the locally nilpotent residual of the join of two subnormal subgroups H and K in a locally finite group, and when H and K satisfy the minimal condition for subnormal subgroups.

In chapter two we shall obtain results similar to (3) for the lower central series when J satisfies the minimal condition on normal subgroups and when H and K satisfy the maximal condition for subnormal subgroups.

Closely tied to these results, is the question of permutability of subnormal subgroups which has been of interest since the 1939 paper. Wielandt [34] showed that for groups with a composition series, a perfect subnormal subgroups permutes with every other subnormal subgroup. In [28] Roseblade extended this result and showed that if the tensor product $H/H' \otimes K/K'$ is trivial and H and K are subnormal subgroups of any group G, then they permute.

Permutability also occurs in (1) and (2) since $H^{\mathfrak{N}}K$,

$H^{s} K$, $H^{r} K^{r}$, $H^{s} K^{s}$ are subgroups.

In [30] Roseblade shows that there always exists some term of the derived series of H that permutes with K. However, it is not known in general whether this is true when derived series is replaced by lower central series. This is certainly true by adapting Wielandt's proof of (1) when H and K satsify the minimal condition for subnormal subgroups. In chapter two, we discuss this question for subnormal subgroups H and K of a group G satisfying the minimal condition for normal subgroups.

Finally, on the question of permutability, we note that in [17] Lennox showed that two subnormal subgroups H and K permute provided that they do so modulo each term of the derived series. Recently it has been shown by Brewster [4] that this remains true when derived series is replaced by lower central series.

In infinite groups, subgroups belonging to series of a group other than a finite series are also considered, in particular ascendant subgroups. A subgroup H of a group G is ascendant in G if there is an ordinal ρ and subgroups $\{ H_\alpha : \alpha \leq \rho \}$ of G such that $H = H_0$, $H_\alpha \lhd H_{\alpha+1}$ $(\alpha < \rho)$

$$H_\mu = \bigcup_{\alpha<\mu} H_\alpha \quad \text{if } \mu \leq \rho \text{ is a limit ordinal and } H_\rho = G.$$

Ascendance was first studied by Plotkin [21] and Gruenberg [6], and even earlier, although not explicitly, in Baer [1]. The join of two ascendant subgroups need not be ascendant, as examples in the case of subnormality show that even the join of two subnormal subgroups need not be ascendant (see for example [31]).

Examples of ascendant coalition classes, $\mathfrak{X}$, (the join of two ascendant $\mathfrak{X}$-subgroups is an ascendant $\mathfrak{X}$-subgroup) are the class of finitely generated nilpotent groups (Gruenberg [6]) and groups with the minimal condition for subnormal subgroups (Robinson [23]).

In general, however, the problem of finding whether a class is an ascendant coalition class seems to be a more difficult problem than deciding whether it is a subnormal coalition class. It is not known whether the classes: finitely generated groups and groups with the maximal condition on subnormal or ascendant subgroups are ascendant coalition classes. Recently Hulse [12] has shown that certain sub-classes of the class of finitely generated groups are ascendant coalition classes.

Results analogous to (1) and (2) for a group G generated by two ascendant subgroups and various questions on permutability of ascendant subgroups have not appeared in the literature. In chapter three of the thesis we discuss these questions when H and K are ascendant subgroups of a group G and H and K satisfy the minimal condition for subnormal subgroups.

Research in subnormality and ascendancy has in recent years been led by Wielandt to examine criteria for subnormality and ascendancy in various classes of groups.

In [36], Wielandt has examined conditions in a finite group G which imply subnormality of a subgroup A. For example if $A_1,\ldots,A_n$ are generating sets of A such that

$$[\,g, A_1,\ldots,A_n\,] \leq A \quad \forall g \in G \qquad (4)$$

then A sn G.

Wielandt has also extended some of his results to groups satisfying the maximal condition, as yet unpublished.

In [10] Hartley and Peng have examined subnormality and ascendancy in groups with the minimal condition for subgroups.

They have shown that if (4) holds for

(i) $n = n(g)$ and (ii) n a fixed integer then in case (i) A is ascendant in G and (ii) A is subnormal in G.

Alternative proofs of results known about left-Engel elements are also obtained in both the cases when G is finite or G satisfies the minimal condition.

In chapter four we extend some of these criteria for subnormality and ascendancy to soluble and generalized soluble groups.

In [20] Peng examines subnormality in soluble-by-Max groups. We shall discuss his results more fully in chapter five, and obtain a criteria for subnormality in soluble groups subject to certain finiteness conditions, which is weaker than those discussed previously.

Notation and Terminology

Capital Roman letters are used to denote groups and small Roman letters for elements of a group. $H \leq G$, $H \trianglelefteq G$ and $H < G$ denote "H is a subgroup of G," "H is a proper subgroup of G," and "H is a normal subgroup of G."

If X is any subset of G, then $\langle X \rangle$ denotes the subgroup of G generated by the elements of X. The normalizer of X in G is denoted by $N_G(X)$ and the centralizer by $C_G(X)$. If Y is a subset of G, we denote the normal closure of X in Y by X^Y and define it as $X^Y = \langle x^y : x \in X, y \in Y \rangle$ where $x^y = y^{-1} x y$.

If $g_1, \ldots, g_n$ are elements of a group G, then $[g_1, g_2] = g_1^{-1} g_2^{-1} g_1 g_2$ and

$$[g_1, \ldots, g_n] = [[g_1, \ldots, g_{n-1}], g_n].$$

If $Y_1, \ldots, Y_n$ are subgroups of G, then

$$[Y_1, Y_2] = \langle [y_1, y_2] ; y_i \in Y_i, \; i = 1,2 \rangle \text{ and}$$

$$[Y_1, \ldots, Y_n] = [[Y_1, \ldots, Y_{n-1}], Y_n].$$

If $Y \leq G$ then $\mathrm{Core}_G Y$ is the largest normal subgroup of G contained in Y.

If p is a prime, then p' denotes the set of primes different from p. Suppose that π is a set of primes then an element x of finite order in a group is said to be a π-element if the prime divisors of the order of x all lie in the set π. A group G is a π-group if every element of G is a π-element.

We shall use and assume the language of group classes and closure operations developed by P. Hall [8].

If $\mathfrak{X}$ is a class of groups then $\mathfrak{X}$ contains all groups of order 1, and if $G \in \mathfrak{X}$ then $\mathfrak{X}$ contains all groups isomorphic to G. If $\mathfrak{X}$ and $\mathfrak{Y}$ are classes of groups, then $\mathfrak{X}\mathfrak{Y}$ is the class defined by : $G \in \mathfrak{X}\mathfrak{Y}$ if and only if G has a normal subgroup N such that $N \in \mathfrak{X}$ and $G/N \in \mathfrak{Y}$.

Let A, B be closure operations. Then $< A, B >$ denotes "the smallest closure operation which contains A and B."

The closure operations which we shall use are S, Sn, Q, P, R_o, R, L, N_o, N', and are defined as follows:

$\mathfrak{X} = s\mathfrak{X}$ if every subgroup of an $\mathfrak{X}$-group is an $\mathfrak{X}$- group.

$\mathfrak{X} = Sn\mathfrak{X}$ if every subnormal subgroup of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group.

$\mathfrak{X} = Q\mathfrak{X}$ if every homomorphic image of an $\mathfrak{X}$-group is an $\mathfrak{X}$-group.

$\mathfrak{X} = P\mathfrak{X}$ if every extension of an $\mathfrak{X}$-group by an $\mathfrak{X}$-group is an $\mathfrak{X}$-group.

$\mathfrak{X} = R_o\mathfrak{X}$ if whenever for normal subgroups N_1, N_2 of G with $G/N_1 \in \mathfrak{X}$, $G/N_2 \in \mathfrak{X}$ then $G/N_1 \cap N_2 \in \mathfrak{X}$.

$\mathfrak{X} = R\mathfrak{X}$ if $N_\lambda \triangleleft G$ and $G/N_\lambda \in \mathfrak{X}$ $(\lambda \in \Lambda)$ always imply $G/\bigcap_{\lambda \in \Lambda} N_\lambda \in \mathfrak{X}$.

$\mathfrak{X} = N_o\mathfrak{X}$ if the product of any pair of normal $\mathfrak{X}$-subgroups is an $\mathfrak{X}$-group.

$G \in L\mathfrak{X}$ or G is locally an $\mathfrak{X}$-group if every finite subset of G lies in an $\mathfrak{X}$-subgroup of G.

$G \in N'\mathfrak{X}$ if G can be generated by its ascendant $\mathfrak{X}$-subgroups.

We denote classes of groups as follows:

$\mathfrak{F}$ = finite groups $\qquad \mathfrak{F}_\pi$ = finite π-groups

$\mathfrak{G}$ = finitely generated groups

$\mathfrak{A}$ = abelian groups

$\mathfrak{N}$ = nilpotent groups

$\mathfrak{S}$ = soluble groups

Max, Max-sn, Max-n = groups with the maximal condition on the set of all subgroups, subnormal subgroups, normal subgroups respectively.

Min, Min-sn, Min-n = groups with the minimal condition on the set of all subgroups, subnormal subgroups, normal subgroups respectively.

Let $\mathfrak{X}$ be a class of groups and let G be any group.

The $\mathfrak{X}$-radical of G is the product of all the normal $\mathfrak{X}$-subgroups of G.

The $\mathfrak{X}$-residual of G is the intersection of all normal subgroups of G whose factor groups in G are $\mathfrak{X}$-groups. We denote the $\mathfrak{X}$-residual of a group G by $G^{\mathfrak{X}}$. $G/G^{\mathfrak{X}}$ is said to be a residually $\mathfrak{X}$-group.

If G is a locally finite group we denote the π-residual, i.e. the intersection of all normal subgroups of G whose factors in G are $L\mathfrak{F}_\pi$-groups, by $O^\pi(G)$.

Series of Subgroups

Let H be a subgroup of a group G and let Σ be a totally ordered set.

A series between H and G of type Σ is a set of subgroups of G $\{U_\sigma, V_\sigma : \sigma \in \Sigma\}$ such that

(i) each U_σ and V_σ contains H.

(ii) $G \setminus H = \bigcup_{\sigma\in\Sigma} (U_\sigma \setminus V_\sigma)$.

(iii) $U_\tau \leq V_\sigma$ if $\tau < \sigma$.

(iv) $V_\sigma \triangleleft U_\sigma$.

A series between 1 and G is called a <u>series in G</u>.

The subgroups U_σ and V_σ are the <u>terms</u> of the series and the groups U_σ / V_σ are the <u>factors</u> of the series. A series in G is called a <u>normal</u> series if the subgroups U_σ and V_σ are all normal subgroups of G.

If the totally ordered set Σ is well-ordered then the series is said to be an <u>ascending</u> series.

In this case we obtain (see Robinson [25] p. 11)

$$H = V_0 \triangleleft V_1 \triangleleft \cdots\cdots V_\sigma \triangleleft V_{\sigma+1} \cdots\cdots V_\alpha = G$$

where $V_\lambda = \bigcup_{\beta<\lambda} V_\beta$ if λ is a limit ordinal

If Σ' denotes Σ with the reverse order $<_*$ i.e. $\sigma <_* \tau$ if and only if $\tau < \sigma$, then if Σ' is well-ordered the series is a <u>descending</u> series.

A subgroup H of a group G is termed <u>serial</u> in G if there is a series between H and G. Should the series be ascending, H is said to be <u>ascendant</u> in G, and if the series has finite length, H is said to be <u>subnormal</u> in G.

We write

$$H \text{ ser } G, \quad H \text{ asc } G, \quad H \text{ sn } G \ .$$

If there is a series between H and G with order type Σ, we write

$$H \triangleleft^{\Sigma} G \ .$$

Let $H \leq G$ and let a descending series of subgroups of G be defined as follows:

$$H^{G,0} = G \qquad H^{G,\alpha+1} = H^{H^{G,\alpha}}$$

and $H^{G,\lambda} = \bigcap_{\alpha<\lambda} H^{G,\alpha}$

for all ordinals α, and for all limit ordinals λ. There exists a first ordinal α such that

$$H^{G,\alpha} = H^{G,\alpha+1}$$

and clearly the series becomes stable at this point,

i.e. $H^{G,\alpha} = H^{G,\beta}$ for all $\beta \geq \alpha$ and

$$H^{H^{G,\alpha}} = H^{G,\alpha} .$$

$\{ H^{G,\beta} : \beta \leq \alpha \}$ is called the <u>normal closure series</u> of H in G.

We write $H \triangleleft^r G$ if there is a series

$$H = H_r \triangleleft H_{r-1} \triangleleft \cdots \triangleleft H_0 = G \text{ with } r \text{ finite.}$$

Let $H \leq G$, then $H \triangleleft^r G$ if and only if $H = H^{G,r}$ (see for example Robinson [22]).

The normal closure series is therefore the fastest descending series of G whose terms all contain H.

If H is subnormal in G, the length of the normal closure series of H in G is called the <u>subnormal index</u>, and is denoted by $s(G{:}H)$.

If $H \triangleleft^r G$ then $s(G{:}H) \leq r$.

If $H, K \leq G$, let

$$\gamma\, HK^n \text{ denote } [H, \underbrace{K, \ldots, K}_{n}] = [H, {}_nK] .$$

For some integer n, $G^n = \langle g^n \mid g \in G \rangle$.

Let $H \leq G$ and let i be a non-negative integer; then $H^{G,i} = H\,{}_{\gamma}^{i}G\,H$ and $H \,{}_{\triangleleft}^{i}\, G$ if and only if ${}_{\gamma}^{i}G\,H \leq H$ (see Robinson [22]).

We denote the n-th terms of the derived series, the lower central series and the upper central series by:

$$G^{(n)}, \quad \gamma_n(G), \quad \zeta_n(G) .$$

We denote the derived subgroup of G by G'. If $G = G'$, G is said to be perfect. $\zeta(G) = \zeta_1(G)$ denotes the centre of G.

The limit of the upper central series of a group G is called the hypercentre of G.

G is said to be a hypercentral group if G coincides with its hypercentre.

G is called a Gruenberg group if G can be generated by its ascendant abelian subgroups.

We shall say a group G has finite (Prüfer) rank, r, if every finitely generated subgroup can be generated by r elements and r is the least positive integer with this property.

An abelian group has finite rank r if and only if

$$r = r_o + \max_p r_p < \infty$$

where the p-component of A is a direct product of r_p cyclic or quasi-cyclic groups $\neq 1$, and the factor group of A with respect to its torsion-subgroup is isomorphic with a subgroup of a direct sum of r_o but no fewer copies of the additive group of rational numbers.

$GL(n,F)$ = General linear group of all invertible nxn matrices over a field F.

Other, more specific definitions are given in the text as they are required.

PART ONE

RESIDUALS AND THE LOWER CENTRAL SERIES OF SUBNORMAL AND ASCENDANT SUBGROUPS

Chapter 1 The Locally Nilpotent Residual of the Join of two Subnormal Subgroups.

Sec. 1 Elementary Results

In this section we prove some elementary general results that we will use later in the thesis. We begin by examining the effect of homomorphisms on the $\mathfrak{X}$ - residual of a group G.

Lemma 1.1 Let $\mathfrak{X} = Q\mathfrak{X}$ and suppose that G is a group such that $G/G^{\mathfrak{X}} \in \mathfrak{X}$. Then if θ is any homomorphism of G, we have $(G\theta)^{\mathfrak{X}} = (G^{\mathfrak{X}})\theta$ and $G\theta/(G\theta)^{\mathfrak{X}} \in \mathfrak{X}$.

Proof Let $N = \operatorname{Ker} \theta$. Then $N \triangleleft G$ and

$$\frac{G/N}{G^{\mathfrak{X}}N/N} \cong G/G^{\mathfrak{X}}N \in Q\mathfrak{X} = \mathfrak{X}.$$

Hence $(G\theta)^{\mathfrak{X}} \leq (G^{\mathfrak{X}})\theta$.

Now let $N \leq L \leq G$ be such that $L/N \triangleleft G/N$ and

$$\frac{G/N}{L/N} \in \mathfrak{X}.$$

Therefore $L \triangleleft G$ and $G/L \cong \frac{G/N}{L/N} \in \mathfrak{X}$.

Hence $G^{\mathfrak{X}} \leq L$ and $G^{\mathfrak{X}}N/N \leq L/N$.

Since this is true for all such L/N we have

$(G^{\mathfrak{X}})\theta \leq (G\theta)^{\mathfrak{X}}$ and hence equality.

Also $G\theta/(G\theta)^{\mathfrak{X}} = G\theta/G^{\mathfrak{X}}\theta \in \mathfrak{X}$.

Therefore $G\theta/(G\theta)^{\mathfrak{X}} \in \mathfrak{X}$.

We next examine the $\mathfrak{X}$ - residual of the join of a subnormal and a normal subgroup.

<u>Lemma 1.2</u> Let $\mathfrak{X} = N_0\mathfrak{X} = S_n\mathfrak{X} = Q\mathfrak{X}$. Suppose that $G = HK$ where H sn G, $K \trianglelefteq G$ and that $K/K^{\mathfrak{X}} \in \mathfrak{X}$, $H/H^{\mathfrak{X}} \in \mathfrak{X}$.

Then $G/G^{\mathfrak{X}} \in \mathfrak{X}$ and $G^{\mathfrak{X}} = H^{\mathfrak{X}}K^{\mathfrak{X}}$.

<u>Proof</u> Let $N \trianglelefteq G$ such that $G/N \in \mathfrak{X}$. Then

$KN/N \in S_n\mathfrak{X} = \mathfrak{X}$.

Since $K/K \cap N \cong KN/N \in \mathfrak{X}$ we have

$K/K \cap N \in \mathfrak{X}$ and so $K^{\mathfrak{X}} \leq K \cap N \leq N$.

Since this is true for any such N we have $K^{\mathfrak{X}} \leq G^{\mathfrak{X}}$. Similarly, $H^{\mathfrak{X}} \leq G^{\mathfrak{X}}$.

Now $K^{\mathfrak{X}}$ char $K \trianglelefteq G \Rightarrow K^{\mathfrak{X}} \trianglelefteq G$. Since $K^{\mathfrak{X}} \leq G^{\mathfrak{X}}$, using lemma 1.1 we may factor G by $K^{\mathfrak{X}}$ and assume $K^{\mathfrak{X}} = 1$.

Hence we must show that $H^{\mathfrak{X}} = G^{\mathfrak{X}}$. By above, $H^{\mathfrak{X}} \leq G^{\mathfrak{X}}$.

Let $H \trianglelefteq^m G$ and use induction on m. If $m = 1$, then $H^{\mathfrak{X}} \trianglelefteq G$ and

$$G/H^{\mathfrak{X}} = \langle H/H^{\mathfrak{X}}, KH^{\mathfrak{X}}/H^{\mathfrak{X}} \rangle$$

where $H/H^{\mathfrak{X}} \in \mathfrak{X}$ and $KH^{\mathfrak{X}}/H^{\mathfrak{X}} \cong K/K \cap H^{\mathfrak{X}} \in Q\mathfrak{X} = \mathfrak{X}$.

So $G/H^{\mathfrak{X}}$ is generated by two normal $\mathfrak{X}$ - subgroups and by N_0 - closure we have $G/H^{\mathfrak{X}} \in \mathfrak{X}$.

Hence $G^{\mathfrak{X}} \leq H^{\mathfrak{X}}$ and we have equality and $G/G^{\mathfrak{X}} \in \mathfrak{X}$.

Now suppose $m > 1$. Let $H_1 = H^K$. Then $H_1 = H_1 \cap HK = H(H_1 \cap K)$ where $H_1 \cap K \in S_n\mathfrak{X} = \mathfrak{X}$.

Since $H \trianglelefteq^{m-1} H_1$, by induction

$$H_1^{\mathfrak{X}} = H^{\mathfrak{X}} \text{ and } H_1/H_1^{\mathfrak{X}} \in \mathfrak{X}.$$

Then, by the case $m = 1$, we have $H_1^{\mathfrak{X}} = G^{\mathfrak{X}}$ and $G/G^{\mathfrak{X}} \in \mathfrak{X}$.

Hence $H_1^{\mathfrak{X}} = H^{\mathfrak{X}} = G^{\mathfrak{X}}$ and the lemma is proved.

Although $G/G^{\mathfrak{X}}$ is always a residually $-\ \mathfrak{X}$ group, it is not necessarily an $\mathfrak{X}$ - group unless G satisfies the minimal condition on normal subgroups, or unless the class $\mathfrak{X}$ is R - closed.

In the case of locally finite groups, however, we have the following known result.

<u>Lemma 1.3</u> Let $\mathfrak{X} = \langle S, R_0 \rangle \mathfrak{X} \leq \mathfrak{F}$.

Then $R L \mathfrak{X} \cap L \mathfrak{F} = L \mathfrak{X}$.

<u>Proof</u> Obviously, $L\mathfrak{X} \leq RL\mathfrak{X} \cap L\mathfrak{F}$. Let $G \in RL\mathfrak{X} \cap L\mathfrak{F}$ Let F be a finitely generated sugbroup of G.

Since $G \in L\mathfrak{F}$ then $F \in \mathfrak{F}$.

Let $1 \neq x \in F$. Since $G \in RL\mathfrak{X}$ there exists a normal subgroup $K(x)$ of G such that $G/K(x) \in L\mathfrak{X}$ and $x \notin K(x)$.

Now $F/F \cap K(x) \cong FK(x)/K(x)$ which is a finite subgroup of the $L\mathfrak{X}$ - group $G/K(x)$.

Hence $FK(x)/K(x)$ is contained in some $\mathfrak{X}$ - group.

Therefore, $FK(x)/K(x) \in S\mathfrak{X} = \mathfrak{X}$.

Now

$$\bigcap_{1 \neq x \in F} (F \cap K(x)) = (1)$$

So, $F \in R_0\mathfrak{X} = \mathfrak{X}$.

Hence $G \in L\mathfrak{X}$.

The following result relates the $L\mathfrak{X}$ - residual of a locally finite group G to the $\mathfrak{X}$ - residuals of its finite subgroups.

Lemma 1.4 Let G be a locally finite group. Let $\mathfrak{X}$ be a class of groups such that $\mathfrak{X} = \langle S, R_0 \rangle \mathfrak{X} \leq \mathfrak{Y}$. Then if $\mathfrak{X}$ is also Q – closed we have

$$G^{L\mathfrak{X}} = \langle F^{\mathfrak{X}} \mid F \text{ is a finite subgroup of } G \rangle .$$

Proof By lemma 1.3 we have that $G/G^{L\mathfrak{X}} \in L\mathfrak{X}$.

Let $R = \langle F^{\mathfrak{X}} \mid F$ is a finite subgroup of $G \rangle$.

Then $R \lhd G$ and if F is any finite subgroup of G

$FR/R \cong F/F \cap R$ which is a homomorphic image of $F/F^{\mathfrak{X}} \in R_0 \mathfrak{X} = \mathfrak{X}$.

Therefore $FR/R \in Q\mathfrak{X} = \mathfrak{X}$.

Now let K/R be any finite subgroup of G/R.

Let $K/R = \langle Rk_1, \ldots, Rk_n \rangle$.

Then $K/R = \langle k_1, \ldots, k_n \rangle R/R \in \mathfrak{X}$ by the above.

Hence $G/R \in L\mathfrak{X}$ and it follows that $G^{L\mathfrak{X}} \leq R$.

However, if F is any finite subgroup of G,

$FG^{L\mathfrak{X}}/G^{L\mathfrak{X}} \in S\mathfrak{X} = \mathfrak{X}$ and $F/F \cap G^{L\mathfrak{X}} \cong FG^{L\mathfrak{X}}/G^{L\mathfrak{X}}$.

Therefore, $F^{\mathfrak{X}} \leq F \cap G^{L\mathfrak{X}} \leq G^{L\mathfrak{X}}$.

So $R \leq G^{L\mathfrak{X}}$ and we have equality.

Sec. 2 Permutability of Locally Nilpotent Residuals

Wielandt has shown in [35], that for a finite group G, generated by two subnormal subgroups H and K

$$G^{\mathfrak{N}} = H^{\mathfrak{N}} K^{\mathfrak{N}} \text{ and } H^{\mathfrak{N}} K = K H^{\mathfrak{N}}.$$

We are concerned here with similar results for the locally nilpotent residual of a group G, generated by two subnormal subgroups, and shall show that permutability occurs when

$$G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle \text{ and } H/H^{L\mathfrak{N}} \in L\mathfrak{N}.$$

As Stonehewer points out in [33], the difficulty in extending the Wielandt proof of permutability of nilpotent residuals is that his proof depends on the fact that for some integer r, defining $H^{\mathfrak{N}^2} = (H^{\mathfrak{N}})^{\mathfrak{N}}$,

$$H^{\mathfrak{N}^i} = (H^{\mathfrak{N}^{i-1}})^{\mathfrak{N}},$$

we have $H^{\mathfrak{N}^r}$ is a perfect subnormal subgroup, and hence permutes with all subnormal subgroups of G (see Wielandt [34]). In general the series

$$H^{\mathfrak{N}} \geq H^{\mathfrak{N}^2} \geq \ldots \geq H^{\mathfrak{N}^i} \geq \ldots$$

will not terminate, and so we do not know whether for some integer r, $H^{\mathfrak{N}^r}$ is perfect.

However, following the method in [33] we use the following generalization by Brewster in [4] of a result by Lennox [17].

<u>Lemma 1.5</u> (Brewster) Suppose that H and K are subnormal subgroups of a group G, that $G = \langle H, K \rangle$ and that for all finite $c \geq 1$, $G = HK\gamma_c(G)$. Then $G = HK$.

A proof of this result is included in the Appendix.

We are now able to generalize theorem A of [33] and obtain:

<u>Theorem A</u> Let $G = \langle H, K \rangle$ where H, K are subnormal subgroups of G, and suppose that

$$G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle. \text{ Then}$$

$$G^{L\mathfrak{N}} = H^{L\mathfrak{N}}K^{L\mathfrak{N}} \text{ provided that } H/H^{L\mathfrak{N}} \in L\mathfrak{N}.$$

<u>Proof</u> Let $M = G^{L\mathfrak{N}}$ and for some integer $c \geq 1$ let $N = \gamma_c(M)$. Hence $M \triangleleft G$. Since the class $L\mathfrak{N}$ satisfies the hypothesis of lemma 1.2 and since by lemma 1.1

$\left(HN/N\right)/\left(HN/N\right)^{L\mathfrak{N}} \in L\mathfrak{N}$, we may apply lemma 1.2

to the product

$$HM/N = (HN/N)(M/N) .$$

Now $M/N \in \mathfrak{N}$ and so $(M/N)^{L\mathfrak{N}} = 1$.

Hence $(HM/N)^{L\mathfrak{N}} = (HN/N)^{L\mathfrak{N}}$

By lemma 1.1 we have $(HN/N)^{L\mathfrak{N}} = H^{L\mathfrak{N}}N/N$

and therefore $(HM/N)^{L\mathfrak{N}} = H^{L\mathfrak{N}}N/N$ and

$H^{L\mathfrak{N}}N/N \lhd HM/N$.

So $H^{L\mathfrak{N}}N \lhd M$ and since $M = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle$ we have

$$M = H^{L\mathfrak{N}}K^{L\mathfrak{N}}\gamma_c(M) .$$

As this is true for all finite $c \geq 1$, by lemma 1.5 we have $M = H^{L\mathfrak{N}}K^{L\mathfrak{N}}$.

<u>Corollary</u> Let $G = \langle H, K \rangle$ where H, K sn G and suppose that

$$G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle .$$

Then if $H/H^{L\mathfrak{N}} \in L\mathfrak{N}$, we have

$$G^{L\mathfrak{N}} = H^{L\mathfrak{N}}K^{L\mathfrak{N}} \text{ and } H^{L\mathfrak{N}}K = KH^{L\mathfrak{N}} ,$$

$HK^{L\mathfrak{N}} = K^{L\mathfrak{N}}H$.

<u>Proof</u> This follows from theorem A and the next lemma.

<u>Lemma 1.6</u> Let $G = \langle H, K \rangle$ and let $\mathfrak{X}$ be a class of groups such that $G^{\mathfrak{X}} = H^{\mathfrak{X}}K^{\mathfrak{X}}$.

Then $H^{\mathfrak{X}}K = KH^{\mathfrak{X}}$ and $K^{\mathfrak{X}}H = HK^{\mathfrak{X}}$.

<u>Proof</u> $KH^{\mathfrak{X}} = KK^{\mathfrak{X}}H^{\mathfrak{X}} = KG^{\mathfrak{X}} = G^{\mathfrak{X}}K$ since $G^{\mathfrak{X}} \lhd G$ and $G^{\mathfrak{X}}K = H^{\mathfrak{X}}K^{\mathfrak{X}}K = H^{\mathfrak{X}}K$.

So $H^{\mathfrak{X}}K = KH^{\mathfrak{X}}$. Similarly $HK^{\mathfrak{X}} = K^{\mathfrak{X}}H$.

<u>Sec. 3</u> <u>The Join Problem</u>

In this section we shall be concerned solely in proving that for a locally finite group G, generated by two subnormal subgroups H and K we have $G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle$. This proof, although it generalizes the methods used by Wielandt [35], is independent of the finite case. In chapter 3, we shall discuss the locally nilpotent and locally soluble residuals of a locally finite group generated by two serial subgroups, and give an alternative proof which reduces to the finite case and the Wielandt theorem

We shall need the following well-known structure theorem for locally nilpotent, locally finite groups.

<u>Lemma 1.7</u> Let $G \in L(\mathfrak{F} \cap \mathfrak{N})$. Then the set G_p of p-elements of G is a subgroup of G, and G is the direct product of the G_p.

<u>Proof</u> See Kurosh [16] Vol II p. 230.

We shall also need

<u>Lemma 1.8</u> (Hirsch - Plotkin - Baer) If $\mathfrak{X} = \langle N_0, S \rangle \mathfrak{X} \leq Max$ then $L\mathfrak{X}$ is N' - closed.

<u>Proof</u> See Robinson [25] p. 57 Theorem 2.31 .

We begin by examining the π - residual $O^{\pi}(G)$ of a locally finite group G. Since the class $\mathfrak{F}_{\pi}$ is S and R_0 - closed we have by lemma 1.3 that $G/O^{\pi}(G)$ is a locally finite π - group.

Lemma 1.9 Let X, Y be subgroups of a locally finite group G such that $G = \langle X, Y \rangle$ and Y sn G. Suppose that every finite subgroup of X lies in a subnormal subgroup of G in X.

If $O^{\pi}(Y) \leq X \cap Y$ then $O^{\pi}(G) \leq X$.

Proof Let $Y \triangleleft^{m} G$ and proceed by induction on m.

Case 1 $m = 1$. Then $Y \triangleleft G$.

By lemma 1.4

$$O^{\pi}(G) = \langle O^{\pi}(F) \mid F \text{ is a finite subgroup of } G \rangle \qquad (*)$$

Now if F is any finite subgroup of G, then F is contained in some finite subgroup F_1 where

$$F_1 = \langle x_1, \ldots, x_n, y_1, \ldots, y_m \rangle \text{ where } x_i \in X,\ y_j \in Y.$$

Therefore $F_1 = \langle X \cap F_1, Y \cap F_1 \rangle$.

By hypothesis there exists a subnormal subgroup X_1 of G such that

$$X \cap F_1 \leq X_1 \leq X.$$

Hence $F_1 = \langle X_1 \cap F_1, Y \cap F_1 \rangle$ and $X_1 \cap F_1$ sn F_1; $Y \cap F_1 \triangleleft F_1$.

Since F_1 is finite $X_1 \cap F_1 / O^{\pi}(X_1 \cap F_1) \in \mathfrak{F}_{\pi}$ and $Y \cap F_1 / O^{\pi}(Y \cap F_1) \in \mathfrak{F}_{\pi}$ and since the class $\mathfrak{F}_{\pi}$ satisfies the hypotheses of lemma 1.2 we have by lemma 1.2

$$O^{\pi}(F_1) = O^{\pi}(X_1 \cap F_1)\, O^{\pi}(Y \cap F_1) \leq X.$$

Therefore $O^{\pi}(F) \leq X$ for all finite subgroups F of G and so by (*)

$$O^{\pi}(G) \leq X \text{ as required.}$$

Case 2 $m > 1$

Suppose by induction that the lemma is true when Y has subnormal defect less than m.

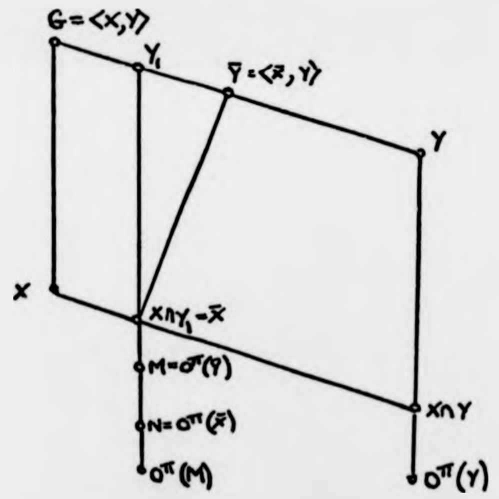
Let $Y_1 = Y^X$. Now if F is any finite subgroup of $Y_1 \cap X$, then by hypothesis there exists a subnormal subgroup S of G such that

$$F \leq S \leq X .$$

Then $F \leq S \cap Y_1 \leq Y_1 \cap X$ and since $Y_1 \lhd G$, $S \cap Y_1 \lhd S$ sn G, i.e. $S \cap Y_1$ sn G.

Hence the group $\overline{Y} = \langle Y_1 \cap X, Y \rangle$ satisfies the hypotheses of the lemma and $Y \lhd^{m-1} \overline{Y}$ since $\overline{Y} \leq Y_1$.

Therefore, by induction we have $O^{\pi}(\overline{Y}) \leq X \cap Y_1$. Let $M = O^{\pi}(\overline{Y})$, $\overline{X} = X \cap Y_1$ and $N = O^{\pi}(\overline{X})$.



Then $M \leq \overline{X}$. Using lemma 1.3 we have

$$\overline{Y}/M \in L\mathfrak{F}_{\pi} \quad \text{and} \quad M/O^{\pi}(M) \in L\mathfrak{F}_{\pi} .$$

Since $O^{\pi}(M)$ char M char $\overline{Y}$ we have $O^{\pi}(M) \triangleleft \overline{X}$.

Therefore, by the P – closure of the class $L\mathfrak{F}_{\pi}$

$$\overline{X}/O^{\pi}(M) \in L\mathfrak{F}_{\pi} \ .$$

Hence $N = O^{\pi}(M)$ char $\overline{Y}$ and so $N = O^{\pi}(\overline{Y})$ and $N \triangleleft G$.

Since $\overline{Y}/N \in L\mathfrak{F}_{\pi}$, $YN/N \in L\mathfrak{F}_{\pi}$.

Now $Y_1/N = \langle YN/N \rangle^G = Y^G N/N$

$$= \langle Y^x N/N \mid x \in X \rangle \ .$$

Since $Y^x N/N = (YN/N)^x \cong YN/N \in L\mathfrak{F}_{\pi}$

we know that Y_1/N is generated by subnormal $L\mathfrak{F}_{\pi}$ – subgroups.
Since the class $\mathfrak{F}_{\pi}$ satisfies the hypotheses of lemma 1.8, $L\mathfrak{F}_{\pi}$ is N'– closed and hence $Y_1/N \in L\mathfrak{F}_{\pi}$.

We have reduced to case 1, applied to the group

$$G/N = \langle X/N, Y_1/N \rangle \text{ since } Y_1/N \triangleleft G/N \ .$$

Hence we may deduce that

$$O^{\pi}(G/N) \leq X/N \ .$$

By lemma 1.1

$$O^{\pi}(G/N) = O^{\pi}(G)N/N \ .$$

Therefore $O^{\pi}(G)N \leq X$ and $O^{\pi}(G) \leq X$.

<u>Lemma 1.10</u> Let G be a locally finite group. Let A, B be subnormal subgroups of G and let $J = \langle A, B \rangle$.

Then if F is any finite subgroup of J, there exists a subnormal subgroup S of G such that

$$F \leq S \leq J \ .$$

Proof This follows immediately from Roseblade and Stonehewer [31] Theorem A.

It is now an easy consequence of lemmas 1.9 and 1.10 to prove

Theorem 1.1 Let G be a locally finite group generated by two subnormal subgroups H and K.

Then $O^{\pi}(G) = \langle O^{\pi}(H), O^{\pi}(K) \rangle$.

Proof Since $H/_{H \cap O^{\pi}(G)} \cong HO^{\pi}(G)/_{O^{\pi}(G)} \in L\mathfrak{F}_{\pi}$

we have $O^{\pi}(H) \le O^{\pi}(G)$. Similarly $O^{\pi}(K) \le O^{\pi}(G)$, and so

$$\langle O^{\pi}(H), O^{\pi}(K) \rangle \le O^{\pi}(G) .$$

By lemma 1.10, the group $\langle O^{\pi}(H), O^{\pi}(K) \rangle$ satisfies the hypotheses of the subgroup X in lemma 1.9 .

Hence in lemma 1.9 let $G = \langle O^{\pi}(H), K \rangle$, $X = \langle O^{\pi}(H), O^{\pi}(K) \rangle$ and $Y = K$. Then we have

$$O^{\pi}(\langle O^{\pi}(H), K \rangle) \le \langle O^{\pi}(H), O^{\pi}(K) \rangle .$$

Again in lemma 1.9 let $G = G$, $X = \langle O^{\pi}(H), K \rangle$ and $Y = H$. The subgroup X satisfies the hypotheses of lemma 1.9 by lemma 1.10 .

Hence we have

$$O^{\pi}(G) \le \langle O^{\pi}(H), K \rangle .$$

Let $L = O^{\pi}(\langle O^{\pi}(H), K \rangle)$.

Then $L \le \langle O^{\pi}(H), O^{\pi}(K) \rangle$ and $L \lhd O^{\pi}(G)$.

Also $O^{\pi}(G)/_{L} \in L\mathfrak{F}_{\pi}$.

Since $O^{\pi}(G)$ char G, we have by lemma 1.3 that

$$O^{\pi}(G)/_{Core_G(L)} \in L\mathfrak{F}_{\pi} .$$

So $O^{\pi}(G) \le L \le \langle O^{\pi}(H), O^{\pi}(K) \rangle$ and hence $O^{\pi}(G) = \langle O^{\pi}(H), O^{\pi}(K) \rangle$.

We now prove the main theorem of this section.

Theorem B Let G be a locally finite group generated by two subnormal subgroups H and K.

Then $G/G^{L\mathfrak{N}}$, $H/H^{L\mathfrak{N}}$, $K/K^{L\mathfrak{N}} \in L\mathfrak{N}$

and $G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle = H^{L\mathfrak{N}} K^{L\mathfrak{N}}$.

Proof By lemma 1.3 we know that $G/G^{L\mathfrak{N}}$, $H/H^{L\mathfrak{N}}$, $K/K^{L\mathfrak{N}}$ are locally nilpotent.

By theorem A we know since $H/H^{L\mathfrak{N}} \in L\mathfrak{N}$ that $H^{L\mathfrak{N}}$ and $K^{L\mathfrak{N}}$ permute, provided that their join is equal to $G^{L\mathfrak{N}}$. Hence it is enough to prove that $G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle$.

Now $H/H \cap G^{L\mathfrak{N}} \cong HG^{L\mathfrak{N}}/G^{L\mathfrak{N}} \in L\mathfrak{N}$ and therefore $H^{L\mathfrak{N}} \leq G^{L\mathfrak{N}}$. Similarly $K^{L\mathfrak{N}} \leq G^{L\mathfrak{N}}$ and so

$$\langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle \leq G^{L\mathfrak{N}} .$$

In theorem 1.1 let $\pi = \{ p \}$ and apply to the group G generated by H and K. Then

$$O^p(G) = \langle O^p(H), O^p(K) \rangle .$$

By lemma 1.7, $K/K^{L\mathfrak{N}}$ is the direct product of p-groups. Hence $O^{p'}(O^p(K)) \leq K^{L\mathfrak{N}}$.

Let $L = O^p(G)$. Applying theorem 1.1 to the group L with $\pi = \{ p' \}$ we have

$$O^{p'}(L) = \langle O^{p'}(O^p(H)), O^{p'}(O^p(K)) \rangle .$$

Now $G^{L\mathfrak{N}} \leq O^p(G)$ and so by lemma 1.3

$$G^{L\mathfrak{N}}/\mathrm{Core}_G \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle \quad \text{is an } L\mathfrak{F}_{p'}\text{-group.}$$

This is true for all primes p.

Hence

$$G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle .$$

<u>Corollary</u> Let $H, K \in$ Min-sn and let H, K be subnormal in their join $G = \langle H, K \rangle$.

Then $G/G^{L\mathfrak{N}}$, $H/H^{L\mathfrak{N}}$, $K/K^{L\mathfrak{N}}$ are locally nilpotent and

$$G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle = H^{L\mathfrak{N}} K^{L\mathfrak{N}} .$$

<u>Proof</u> By the subnormal coalescence of the class Min-sn (see for example [23]) we know that $G \in$ Min-sn .

Therefore $G/G^{L\mathfrak{N}}$, $K/K^{L\mathfrak{N}}$, $H/H^{L\mathfrak{N}}$ are locally nilpotent, since H, K and G satisfy Min-sn.

Now $H/H^{L\mathfrak{N}}$, $K/K^{L\mathfrak{N}}$ are locally nilpotent groups satisfying Min-sn and therefore they satisfy Min and are soluble (see Robinson [25] p. 154).

By the condition Min-sn we have $G/G^{\mathfrak{S}}$, $H/H^{\mathfrak{S}}$, and $K/K^{\mathfrak{S}}$ are soluble and so by Stonehewer [32] we have

$$G^{\mathfrak{S}} = H^{\mathfrak{S}} K^{\mathfrak{S}} \leq H^{L\mathfrak{N}} K^{L\mathfrak{N}} .$$

Hence, without loss of generality, we may assume $G^{\mathfrak{S}} = 1$, and G is soluble. Since G satisfies Min-sn, G is Černikov and locally finite.

Hence $G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle = H^{L\mathfrak{N}} K^{L\mathfrak{N}}$ by theorem B.

Chapter 2. The Lower Central Series of the Join of two Subnormal Subgroups.

Sec. 1 The Minimal Condition on Normal Subgroups.

We consider the nilpotent residual of a group G, generated by two subnormal subgroups H and K, which has the minimal condition on normal subgroups. The condition Min-n of course ensures that the nilpotent residual of G has nilpotent factor group. However, since the condition Min-n is not necessarily inherited by normal subgroups of G (See Robinson [26] p. 153), we have no guarantee that $H/H^{\mathfrak{N}}$ and $K/K^{\mathfrak{N}}$ are nilpotent.

However, we are able to prove more generally:

Theorem C Let $G \in$ Min-n and let H and K be subnormal subgroups of G such that $G = \langle H, K \rangle$. Then for any positive integers r_1, r_2 we have that

$$G^{\mathfrak{N}} \leq \gamma_{r_1}(H)\,\gamma_{r_2}(K)$$

For the proof we shall need the following result on the derived series of the join of two subnormal subgroups, given in [30].

Lemma 2.1 (Roseblade) Let H and K be subnormal subgroups of a group $G = \langle H, K \rangle$ and let α, β be any positive integers. Then there exists an integer δ such that

$$G^{(\delta)} \leq H^{(\alpha)} K^{(\beta)}$$

We shall also require the following well-known results on groups satisfying Min-n.

<u>Lemma 2.2</u> (Baer) A soluble group which satisfies Min-n is locally finite.

<u>Proof</u> See Robinson [25] p. 153.

<u>Lemma 2.3</u> (Čarin, Mc Lain) The locally nilpotent groups which satisfy Min-n are precisely the hypercentral Černikov groups. Thus for locally nilpotent groups, the properties Min and Min-n coincide.

<u>Proof</u> See Robinson [25] p. 154.

<u>Lemma 2.4</u> (Baer) G is a nilpotent group satisfying Min if and only if $\zeta(G)$ satisfies Min and $G/\zeta(G)$ is a finite nilpotent group.

<u>Proof</u> See Robinson [25] p. 69.

Wielandt has shown in [36], for groups G with a composition series, and generated by two subnormal subgroups H and K, that once the join property $G^{\mathfrak{n}} = \langle H^{\mathfrak{n}}, K^{\mathfrak{n}} \rangle$ is established for groups in this class, then we have permutability of the residuals, i.e. $G^{\mathfrak{n}} = H^{\mathfrak{n}}K^{\mathfrak{n}}$.

Wielandt's argument can be adapted to groups in the class Min-sn, and this we prove in lemma 2.5.

Following Wielandt, we define $G^{\mathfrak{n}^r}$ inductively by: $G^{\mathfrak{n}^2} = (G^{\mathfrak{n}})^{\mathfrak{n}}$

$$G^{\mathfrak{n}^r} = (G^{\mathfrak{n}^{r-1}})$$

<u>Lemma 2.5</u> Suppose that for a group G, and subnormal subgroups of G, satisfying the minimal condition on subnormal subgroups and generated by two subnormal subgroups H and K, that $G^{\mathfrak{n}} = \langle H^{\mathfrak{n}}, K^{\mathfrak{n}} \rangle$. Then $H^{\mathfrak{n}}$ permutes with $K^{\mathfrak{n}}$ and so $G^{\mathfrak{n}} = H^{\mathfrak{n}}K^{\mathfrak{n}}$.

<u>Proof</u> Since H is subnormal in G, H satisfies the minimal condition on subnormal subgroups.

Hence for r sufficiently large

$$H^{\mathfrak{N}^r} = H^{\mathfrak{N}^{r+1}}$$

and $H^{\mathfrak{N}^r}$ is a perfect subnormal subgroup of G. By [28] we obtain that $H^{\mathfrak{N}^r}$ permutes with all subnormal subgroups of G.

In particular $K^{\mathfrak{N}}H^{\mathfrak{N}^r}$ is a subgroup.

Now suppose that for any integer i, that $K^{\mathfrak{N}}H^{\mathfrak{N}^i}$ is a subgroup. Then

$$K^{\mathfrak{N}}H^{\mathfrak{N}^{i-1}} = K^{\mathfrak{N}}H^{\mathfrak{N}^i}\, H^{\mathfrak{N}^{i-1}}$$

Now $H^{\mathfrak{N}^{i-1}}$, $K \in$ Min-sn and since Min-sn is a subnormal coalition class [23], the group $< K, H^{\mathfrak{N}^{i-1}} > \in$ Min-sn, and is subnormal in G.

By hypothesis $< K, H^{\mathfrak{N}^{i-1}} >^{\mathfrak{N}} = < K^{\mathfrak{N}}, H^{\mathfrak{N}^i} >$.

Hence, $K^{\mathfrak{N}}H^{\mathfrak{N}^{i-1}} = < K, H^{\mathfrak{N}^{i-1}} >^{\mathfrak{N}} H^{\mathfrak{N}^{i-1}}$ and since

$< K, H^{\mathfrak{N}^{i-1}} >^{\mathfrak{N}}$ is normalized by $H^{\mathfrak{N}^{i-1}}$ we have that $K^{\mathfrak{N}}H^{\mathfrak{N}^{i-1}}$ is a subgroup. By induction on i decreasing we obtain $K^{\mathfrak{N}}H^{\mathfrak{N}}$ is a subgroup.

We are now able to prove theorem C.

<u>Proof of theorem C.</u> We know that there exist positive integers α and β such that

$$H^{(\alpha)} \le \gamma_{r_1}(H) \quad \text{and} \quad K^{(\beta)} \le \gamma_{r_2}(K)$$

So by lemma 2.1 there exists an integer δ such that $G^{(\delta)} \le \gamma_{r_1}(H)\,\gamma_{r_2}(k)$.

Now $\gamma_{r_1}(HG^{(\delta)}/G^{(\delta)}) = \gamma_{r_1}(H)G^{(\delta)}/G^{(\delta)}$

and similarly

$$\gamma_{r_2}\left(HG^{(\delta)}/G^{(\delta)} \right) = \gamma_{r_2}(K)\, G^{(\delta)}/G^{(\delta)}$$

Hence, without loss of generality, we may assume $G^{(\delta)} = 1$ and that G is soluble. By lemma 2.2, G is locally finite. By theorem B we know that the factors $G/G^{L\mathfrak{N}}$, $H/H^{L\mathfrak{N}}$, $K/K^{L\mathfrak{N}}$ are locally nilpotent and that

$$G^{L\mathfrak{N}} = H^{L\mathfrak{N}} K^{L\mathfrak{N}} \leq \gamma_{r_1}(H)\, \gamma_{r_2}(K)$$

As before, we may assume $G^{L\mathfrak{N}} = 1$ so that G is locally nilpotent. But by lemma 2.3 G satisfies the minimal condition. Hence $H/H^{\mathfrak{N}}$ and $K/K^{\mathfrak{N}}$ are nilpotent and

$$H^{\mathfrak{N}} K^{\mathfrak{N}} \leq \gamma_{r_1}(H)\, \gamma_{r_2}(K)\ .$$

We wish to show that $G^{\mathfrak{N}} = \langle H^{\mathfrak{N}}, K^{\mathfrak{N}} \rangle$ for then by lemma 2.5 we know that $G^{\mathfrak{N}} = H^{\mathfrak{N}} K^{\mathfrak{N}}$.

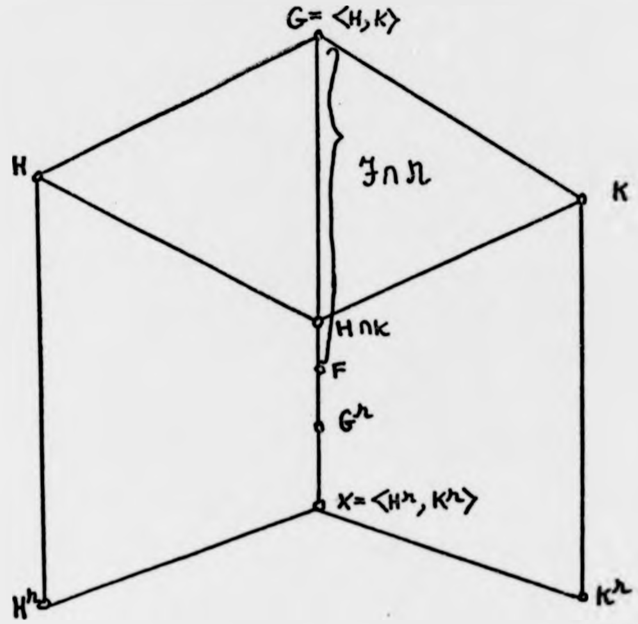
Let $X = \langle H^{\mathfrak{N}}, K^{\mathfrak{N}} \rangle$.

Then $X \leq G^{\mathfrak{N}}$ since $H/H \cap G^{\mathfrak{N}} \cong HG^{\mathfrak{N}}/G^{\mathfrak{N}} \in \mathfrak{N}$

$\Rightarrow$ $H^{\mathfrak{N}} \leq G^{\mathfrak{N}}$, and similarly $K^{\mathfrak{N}} \leq G^{\mathfrak{N}}$.

Now by lemma 2.3, G is a Černikov group, i.e. an extension of an abelian group satisfying Min by a finite group. Let F be the finite residual of G. Then F is abelian and G/F is a finite nilpotent group.

Now by lemma 1.2 $(HF)^{\mathfrak{N}} = H^{\mathfrak{N}} F^{\mathfrak{N}} = H^{\mathfrak{N}}$ since F is abelian. Similarly $(KF)^{\mathfrak{N}} = K^{\mathfrak{N}}$. Hence without loss of generality we may assume that $F \leq H \cap K$. Also since G/F is nilpotent we have $G^{\mathfrak{N}} \leq F$.



Consider $H/H^{\mathfrak{N}}$. The finite residual of $H/H^{\mathfrak{N}}$ is $F/H^{\mathfrak{N}}$. Since $H/H^{\mathfrak{N}}$ is nilpotent, we have by lemma 2.4 that

$F/H^{\mathfrak{N}} \le \zeta(H/H^{\mathfrak{N}})$. So every subgroup $L/H^{\mathfrak{N}}$ contained in $F/H^{\mathfrak{N}}$ is centralized by H, and so H normalizes every subgroup L such that $H^{\mathfrak{N}} \le L \le F$. In particular, H normalizes X. Similarly K normalizes X. Hence $X \lhd G$.

Now $G/X = \langle H/X, K/X \rangle$

Since $H/X, K/X \in \mathfrak{N}$, G/X is generated by two subnormal nilpotent subgroups and G/X satisfies Min. Hence by [23] we have that $G/X \in \mathfrak{N}$ and so $G^{\mathfrak{N}} \le X$.

Hence $G^{\mathfrak{N}} = \langle H^{\mathfrak{N}}, K^{\mathfrak{N}} \rangle$ and the theorem follows since $G^{\mathfrak{N}} = H^{\mathfrak{N}} K^{\mathfrak{N}} \le \gamma_{r_1}(H)\gamma_{r_2}(K)$.

We introduce some notation before proving a corollary to the theorem. If H and K are subgroups of a group G, then the permutizer, $P_H(K)$, of K in H is defined in [31] to be the largest subgroup of H which permutes with K.

Corollary Let $G \in$ Min-n and let $G = \langle H, K \rangle$ where H and K are subnormal subgroups of G. Then there exists a positive integer δ such that $\gamma_\delta(H) \leq P_H(K)$.

Proof We have that $G^{n} \leq HK$ from theorem C. Hence if $L = G^{n}K$, then $L = (H \cap L)K$, and since L is a subgroup we have that $H \cap L$ permutes with K. Hence $H \cap L \leq P_H(K)$. Since $\gamma_\delta(H) \leq G^{n} \cap H$ for some δ we have that $\gamma_\delta(H) \leq H \cap L$.

Hence $\gamma_\delta(H) \leq P_H(K)$.

We note that the proof of the corollary is taken from that given in the case of the derived series in [30] (see [30] corollary to Theorem B).

Sec. 2 The Maximal Condition on Subnormal Subgroups.

We begin with an important theorem of Mal'cev which relates subgroups of a polycyclic group to the subgroups of finite index that contain them. Although the Mal'cev theorem is true for a slightly larger class of groups than polycyclic groups, we shall only be concerned here with the polycyclic case. We shall need this theorem several times in the thesis and since it is not readily accessible in the literature we give a proof here based on that given by Mal'cev in [18].

Theorem 2.1 (Mal'cev) Let G be a polycyclic group. Then any subgroup of G is equal to the intersection of all the subgroups of finite index that contain it.

<u>Proof</u> We use induction on d, the derived length of G. If $d = 1$, then G is abelian. Hence every subgroup of G is a normal subgroup of G, and since each factor group is residually finite, the theorem follows.

So assume $d > 1$ and let G_1 be the derived group of G. Let H be any subgroup of G and let F be the intersection of all subgroups of finite index containing it. Then obviously $H \leq F$. Suppose for a contradiction that $H < F$. Hence we can find an element $f \in F$ such that $f \notin H$. We consider two cases.

<u>Case 1.</u> $f \notin H G_1$. Since f is contained in the intersection of subgroups of finite index containing $H G_1$, we may, without loss of generality, factor G by G_1 and assume $G_1 = 1$ and G is abelian. But then by the case $d = 1$, we obtain $f \in H$, a contradiction.

<u>Case 2.</u> $f \in H G_1$. Then $f = h_1 g_1 \quad h_1 \in H, g_1 \in G_1$. Let $T = H \cap G_1$. Then $g_1 \notin T$ and G_1 has derived series of length $d - 1$.

Hence by induction there exists a subgroup F_1 of finite index in G_1, containing T such that $g_1 \notin F_1$.

Let $|G_1 : F_1| = s$. Then $G_1^{s!} \leq F_1$ and $G_1^{s!}$ char $G_1 \triangleleft G$, so $G_1^{s!} \triangleleft G$ and $g_1 \notin T G_1^{s!}$ (1).

Suppose that $f \in H G_1^{s!}$. Then $f = h_2 g_2$ where $h_2 \in H$, $g_2 \in G_1^{s!}$, and $h_2 g_2 = h_1 g_1$.

So $h_1^{-1} h_2 = g_1 g_2^{-1} \in H \cap G_1 = T$.

So $g_1 \in H G_1^{s!}$ which is a contradiction to (1). So $f \notin H G_1^{s!}$.

Hence, without loss of generality, we may factor G by $G_1^{s!}$ and assume $G_1 \in \mathcal{F}$ and $G_1^{s!} = 1$.

So $|HG_1 : H| = |G_1 : H \cap G_1| = t$, say.

Let $M = HG_1$. Then $M^{t!} \leq H$. So $f \notin M^{t!}$ and we may assume $M^{t!} = 1$ and M is finite.

Then H is finite, of order n, say. Since G is residually finite, for each $h_i \in H$ $i = 1, \ldots, n$ there exists a normal subgroup N_i of G such that $|G : N_i| < \infty$ and $fN_i \neq h_i N_i$.

Let $N = \bigcap_{i=1}^{n} N_i$. Then $|G : N|$ is finite and $f \notin HN$. But $H \leq HN$ and $|G : HN|$ is finite, a contradiction.

We also use the following lemma.

<u>Lemma 2.6</u> Let $G = HN$ where $N \lhd G$ and H is subnormal in G. Then, given any positive integers r_1, r_2 there exists an integer r such that

$$\gamma_r(G) \leq \gamma_{r_1}(H)\, \gamma_{r_1}(N).$$

<u>Proof</u> Since $N \lhd G$, $\gamma_{r_2}(N)$ is a normal subgroup of G and hence without loss of generality we may assume $\gamma_{r_2}(N) = 1$ and that N is nilpotent.

Suppose that the normal closure series of H in G is

$$H = H_n \lhd H_{n-1} \lhd \cdots \lhd H_1 \lhd H_o = G.$$

Now for each i, $H_{i-1} \lhd H_i$ and $H_i = H_i \cap H_{i-1}N = H_{i-1}(H_i \cap N$

Let $i = n-1$. Then by Fitting's theorem we obtain that

$H_{n-1}/\gamma_{r_1}(H)$ is nilpotent.

Hence there exists an integer c_{n-1} such that

$$\gamma_{c_{n-1}}(H_{i-1}) \leq \gamma_{r_1}(H).$$

Suppose that we have found an integer c_{i-1} such that

$$\gamma_{c_{i-1}}(H_{i-1}) \leq \gamma_{r_1}(H).$$

Then by Fitting's theorem $H_i / \gamma_{c_{i-1}}(H_{i-1})$ is nilpotent.

Hence there exists an integer c_i such that

$$\gamma_{c_i}(H_i) \leq \gamma_{c_{i-1}}(H_{i-1}) \leq \gamma_{r_1}(H).$$

By induction on i decreasing we obtain an integer $c_o = r$ such that

$\gamma_r(G) \leq \gamma_{r_1}(H)$ and the lemma is proved.

We shall now prove the following theorem on the join of two subnormal subgroups which satisfy the maximal condition on subnormal subgroups.

<u>Theorem D.</u> Suppose that H and K are subnormal subgroups of their join $G = \langle H, K \rangle$, and that H and K satisfy the maximal condition for subnormal subgroups. Then, given any two positive integers r_1 and r_2, there exists a positive integer r such that

$$\gamma_r(G) \leq \langle \gamma_{r_1}(H), \gamma_{r_2}(K) \rangle .$$

<u>Proof</u> By the subnormal coalescence of the class Max-sn, [29], we have that $G \in$ Max-sn. Now there exist positive integers s_1, s_2 such that

$$H^{(s_1)} \leq \gamma_{r_1}(H) \qquad K^{(s_2)} \leq \gamma_{r_2}(K) .$$

Then by lemma 2.1 there exists a positive integer s such that

$$G^{(s)} \leq H^{(s_1)} K^{(s_2)} \leq \langle \gamma_{r_1}(H), \gamma_{r_2}(K) \rangle .$$

Let $X = \langle \gamma_{r_1}(H), \gamma_{r_2}(K) \rangle$. Now, without loss of generality, we may assume $\mathrm{Core}_G(X) = 1$. Hence $G^{(s)} = 1$ and

G is soluble. Since $G \in$ Max-sn, G is polycyclic.

Let G have derived length d. We use induction on d. If $d = 1$, then G is abelian and the theorem is trivially true. So assume $d > 1$ and that the theorem is true for soluble groups of derived length less than d. Let N be the last but one term of the derived series. Then G/N has derived length $d-1$, and by induction there exists a positive integer r_3 such that $\gamma_{r_3}(G) \leq NX$.

By lemma 2.6 there exist positive integers t_1, t_2 such that

$$\gamma_{t_1}(NH) \leq \gamma_{r_1}(H) \quad \text{and} \quad \gamma_{t_2}(NK) \leq \gamma_{r_2}(K) .$$

So replacing r_1, r_2 by t_1, t_2 respectively, we may assume that $N \leq H \cap K$. Hence N normalizes X and so $X \triangleleft XN$.

Now $XN/X \cong N/N \cap X \in \mathfrak{A}$

Let $L = \gamma_{r_3}(G)$.

Then $L' \leq (XN)' \leq X$, and since $L' \triangleleft G$ we have $L' = 1$ since $\mathrm{Core}_G(X) = 1$.

Let $M = XL$. Then $X \triangleleft M$ and $M/X \cong L/L \cap X \in \mathfrak{A}$

Since M is polycyclic, there exists $F \triangleleft M$, $X \leq F$, such that M/F is finite and F/X is torsion-free. Let $|M : F| = n$. Then $M^n \leq F$, so we have $|M : M^nX|$ is finite and M^nX/X is torsion-free.

Now by theorem 2.1, there exists a subgroup F_0 of G such that $|G : F_0| < \infty$ and

$$M^nX = F_0 \cap M.$$

By Wielandt [35] we have that there exists an integer r_4 such that

$$\gamma_{r_4}(G) \leq X\,\mathrm{Core}_G(F_0) .$$

Let $R = \mathrm{Core}_G(F_o)$ and let $r_5 = \max\{ r_3, r_4 \}$, then

$$\gamma_{r_5}(G) \leq XR \cap M$$

$$= X(R \cap M) \leq X(F_o \cap M) = XM^n .$$

Hence we may suppose that $L = \gamma_{r_5}(G)$ and that M/X is torsion-free.

By the same method we can now show that for all primes p, there exist positive integers $r(p)$ such that

$$\gamma_{r(p)}(G) \leq XM^p .$$

Let l be the rank of G.

Then M/XM^p has order less than or equal p^l.

Since $L = \gamma_{r_5}(G)$ we have

$\gamma_{r_5+1}(G) \leq XM^p$ for all primes p.

Hence $\gamma_{r_5+1}(G) \leq \bigcap_p XM^p$.

Now $(M/X)^p = XM^p/X$ and since M/X is torsion-free abelian of finite rank

$$\bigcap (M/X)^p = X .$$

Hence $\bigcap_p XM^p = X$ and $\gamma_{r_5+1}(G) = X$.

Let $r = r_5 + 1$. Then

$\gamma_r(G) \leq \langle \gamma_{r_1}(H), \gamma_{r_2}(K) \rangle$ and the theorem is proved.

Chapter 3 Residuals of the Join of two Ascendant Subgroups.

Sec. 1 Locally Finite Groups.

We begin with an alternative proof to theorem B, which also works for serial subgroups. The proof, however, reduces to the finite case and the Wielandt theorem given in [35] .

Theorem 3.1 Let $G = \langle A, B \rangle$ be a locally finite group, where A and B are serial subgroups of G. Let $\mathfrak{X} = \langle R_0, S, Q \rangle \mathfrak{X}$ and suppose that for any two finite subgroups X and Y which are subnormally embedded in their join that

$$\langle X, Y \rangle^{\mathfrak{X}} = \langle X^{\mathfrak{X}}, Y^{\mathfrak{X}} \rangle = X^{\mathfrak{X}} Y^{\mathfrak{X}}$$

Then $G/G^{L\mathfrak{X}} \in L\mathfrak{X}$ and $G^{L\mathfrak{X}} = \langle A^{L\mathfrak{X}}, B^{L\mathfrak{X}} \rangle = A^{L\mathfrak{X}} B^{L\mathfrak{X}}$.

Proof By lemma 1.3 we have $G/G^{L\mathfrak{X}} \in L\mathfrak{X}$.

Let $N = G^{L\mathfrak{X}}$ and $M = \langle A^{L\mathfrak{X}}, B^{L\mathfrak{X}} \rangle$. We will show first that $M = N$.

Now $A/A \cap N \cong AN/N \in L(\mathfrak{F} \cap \mathfrak{X})$

So $A^{L\mathfrak{X}} \leq A \cap N \leq N$. Similarly $B^{L\mathfrak{X}} \leq N$.
So $M \leq N$.

By lemma 1.4 $N = \langle F^{\mathfrak{X}} \mid F$ is a finite subgroup of $G \rangle$. So it is enough to prove that $F^{\mathfrak{X}} \leq M$ for finite subgroups F of G.

Let F be any finite subgroup of G. Then F is contained in some subgroup F_1 where

$$F_1 = \langle a_1, \ldots, a_n, b_1, \ldots, b_m \rangle \quad a_i \in A \quad b_j \in B .$$

Then F_1 is finite and

$$F_1 = \langle F_1 \cap A, F_1 \cap B \rangle .$$

So $F^{\mathfrak{X}} \le F_1^{\mathfrak{X}} = \langle F_1 \cap A, F_1 \cap B \rangle^{\mathfrak{X}}$.

Since A is a serial subgroup of G, $A \cap F_1$ is subnormal in the finite subgroup F_1. Similarly $B \cap F_1$ is subnormal in F_1.

So, by hypothesis

$$
\begin{aligned}
F^{\mathfrak{X}} &\le \langle A \cap F_1, B \cap F_1 \rangle^{\mathfrak{X}} \\
&= \langle (A \cap F_1)^{\mathfrak{X}}, (B \cap F_1)^{\mathfrak{X}} \rangle \\
&\ddot{\le} M
\end{aligned}
$$

So $N \le M$ and we have

$$G^{L\mathfrak{X}} = \langle A^{L\mathfrak{X}}, B^{L\mathfrak{X}} \rangle .$$

To show permutability of $A^{L\mathfrak{X}}$ and $B^{L\mathfrak{X}}$ let $a \in A^{L\mathfrak{X}}$ and $b \in B^{L\mathfrak{X}}$.

Then $ab \in G^{L\mathfrak{X}}$ and as before

$$G^{L\mathfrak{X}} = \langle F^{\mathfrak{X}} \mid F \text{ is a finite subgroup of } G \rangle .$$

So $ab \in E_1^{\mathfrak{X}} \ldots E_s^{\mathfrak{X}}$ where E_i is finite $1 \le i \le s$.

Let $E = \langle E_1 \ldots E_s \rangle$.

Then $ab \le E^{\mathfrak{X}}$ and E is finite, and is contained in some finite subgroup E_o, such that $E_o = \langle E_o \cap A, E_o \cap B \rangle$.

Hence $ab \in E^{\mathfrak{X}} \ddot{\le} E_o^{\mathfrak{X}}$

$$
\begin{aligned}
&= \langle A \cap E_o, B \cap E_o \rangle^{\mathfrak{X}} \\
&= (A \cap E_o)^{\mathfrak{X}} (B \cap E_o)^{\mathfrak{X}} \\
&= (B \cap E_o)^{\mathfrak{X}} (A \cap E_o)^{\mathfrak{X}} \\
&\ddot{\le} B^{L\mathfrak{X}} A^{L\mathfrak{X}} .
\end{aligned}
$$

Hence $ab \in B^{L\mathfrak{X}} A^{L\mathfrak{X}}$. So $B^{L\mathfrak{X}} A^{L\mathfrak{X}} = A^{L\mathfrak{X}} B^{L\mathfrak{X}}$.

Sec. 2 Groups with the Minimal Condition on Subnormal Subgroups.

Let $G \in$ Min-sn. Let $F(G)$ be the smallest subgroup of finite index of the group G. Let $E(G) = F(G)'$. Then $E(G)$ is the smallest subgroup with Cernikov factor group. The following lemma is proved by Hartley and Peng in [10]. We include their proof as it has not yet appeared in the literature.

Lemma 3.1 Let $H, K \in$ Min-sn and suppose that H and K are ascendant subgroups of a group G. Then $E(H) \leq N_G(K)$.

Proof Suppose for a contradiction that the result is false so that there exist ascendant subgroups H_o, K_o of a group G_o which satisfy Min-sn but are such that $H_1 = E(H_o)$ does not normalize K_o. Among the subnormal subgroups of K_o which are not normalized by H_1, let K_1 be minimal, and let $G_1 = \langle H_1, K_1 \rangle$. By construction, H_1 normalizes every proper subgroup of K_1, and hence normalizes also their product L. Therefore $L < K_1$, and clearly $L \triangleleft G_1$.

Letting $G = G_1/L$, $H = H_1L/L$, $K = K_1L/L$

we now have, using the fact that for any group G, $E(E(G)) = E(G)$ (since the class of Cernikov groups is extension closed)

H, K asc G (1)

$H = E(H)$ (2)

K is simple (3)

$H \nleq N_G(K)$ (4)

From Robinson ([23] Lemma 4.3 and Corollary), we obtain

$H \triangleleft^2 G$ (5)

If $K = K'$, then it follows from (1) and (3) that $K \triangleleft^2 G$. This follows by a simple induction on the length of a series connecting K to G, using the well-known fact that a non-abelian simple subnormal subgroup of a group is at most 2 - step subnormal. But then $H \leq N_G(K)$ since $F(G)$ normalizes every subnormal subgroup (see [27]). Hence (4) is contradicted.

Therefore K is cyclic of prime order p, and the normal closure $\overline{K} = K^G$ is a locally finite p - group, by (1) and [25] p. 20 according to which the class of locally finite p - groups is closed under forming joins of ascendant subgroups. Now from (5) we have $[\overline{K}, H] \leq \overline{K} \cap H^G = R$ and $[R, H] \leq \overline{K} \cap H = S$, a normal locally finite p - subgroup of H. Since H satisfies Min-sn, so does S, and hence S is Černikov ([25] p. 171). Now $S/F(S)$ is finite and $F(S)$ is the union of its finite characteristic subgroups; since $S \lhd H$ and H has no proper subgroups of finite index, it follows that H centralizes $S/F(S)$ and $F(S)$. Therefore, $H' = H$ centralizes S.

Hence H stabilizes by conjugation the series

$$1 \lhd S \lhd R \lhd \overline{K}$$

of $\overline{K}$ and we find successively that $H' = H$ centralizes R and then $\overline{K}$. We now have $H \leq N_G(K)$, a final contradiction to (4).

We examine $E(G)$ in the following case:

<u>Lemma 3.2</u> Let $G = \langle H, K \rangle$ where H, K asc G and $H, K \in$ Min-sn. Then $G \in$ Min-sn and

$$E(G) = E(H)E(K).$$

<u>Proof</u> The class Min-sn forms a subnormal and an ascendant coalition class (see [23] theorems 4.1 and 4.2). Hence $G \in$ Min-sn.

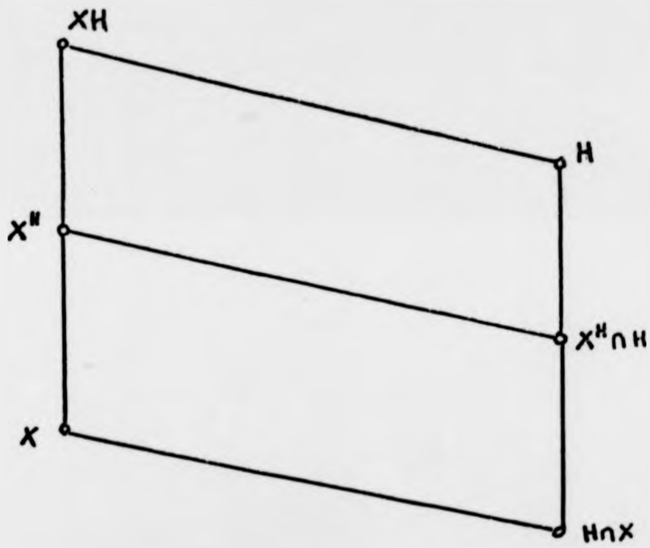
Now $E(H) \lhd H$ and so $E(H) \in$ Min-sn; similarly $E(K) \in$ Min-sn.

Since $E(H)$ and $E(K)$ have no proper subgroups of finite index, they are subnormal in G, and normalize each other ([23] lemma 4.3). Hence from above $E(H)E(K)$ is a subnormal subgroup of G satisfying Min-sn.

Let $X = E(H)E(K)$.

Then $E(H) \stackrel{\cdot\cdot}{\leq} E(G)$ and $E(K) \stackrel{\cdot\cdot}{\leq} E(G)$. Hence $X \stackrel{\cdot\cdot}{\leq} E(G)$, and $X \triangleleft E(G)$, by lemma 3.1. Hence $X \triangleleft^2 G$.

We consider $L = \langle H, E(K) \rangle$. Then by lemma 3.1 $E(G) \stackrel{\cdot\cdot}{\leq} N_G(H)$ and hence $X \stackrel{\cdot\cdot}{\leq} N_G(H)$. So $L = XH$.



Now $X \triangleleft^2 XH$.

Then $XH/X^H = X^H H / X^H \cong H / X^H \cap H$

which is Černikov.

Also $X^H = X^H \cap XH$

$= X(X^H \cap H)$

and therefore $X^H/X \cong X^H \cap H / X \cap H$ which is Černikov.

Since $|XH : N_{XH}(X)| < \infty$ (see [21] or [23]) and by the P-closure of the class of Černikov groups ([25] p.69)

$$XH / \mathrm{Core}_{XH}(X) \quad \text{is Černikov.}$$

Hence $E(XH) \leq X$.
But $E(H) \leq E(XH)$ and $E(K) = E(E(K)) \leq E(XH)$.
Therefore $E(XH) = X$ and X is normalized by H. Similarly X is normalized by K. So $X \triangleleft G$ and

$$G/X = \langle HX/X, KX/X \rangle$$

is generated by two ascendant Černikov groups, and so is Černikov by [23] theorem 4.2.

Hence $E(G) \leq X$ and so

$E(G) = E(H)E(K)$.

We are now able to prove:

<u>Theorem E</u> Let $G = \langle H, K \rangle$ where $H, K \in$ Min-sn and H, K asc G. Then $G/G^{\mathfrak{S}} \in \mathfrak{S}$ and

$$G^{\mathfrak{S}} = \langle H^{\mathfrak{S}}, K^{\mathfrak{S}} \rangle = H^{\mathfrak{S}} K^{\mathfrak{S}}.$$

<u>Proof</u> Since Min-sn is an ascendant coalition class we have $G \in$ Min-sn. Hence $G/G^{\mathfrak{S}} \in \mathfrak{S}$.

Let $A = G^{\mathfrak{S}}$ and $B = \langle H^{\mathfrak{S}}, K^{\mathfrak{S}} \rangle$.

Since, as before $H^{\mathfrak{S}} \leq G^{\mathfrak{S}}$ and $K^{\mathfrak{S}} \leq G^{\mathfrak{S}}$, we therefore have $B \leq A$.

Now $G/E(G)$ is Černikov, and hence is locally finite.

Now $(G/E(G))^{L\mathfrak{S}} = (G/E(G))^{\mathfrak{S}}$

and by the Q-closure of the class $\mathfrak{S}$ we have

$$(G/E(G))^{\mathfrak{s}} = G^{\mathfrak{s}}E(G)/E(G) \quad \text{by lemma 1.1 .}$$

Since $G/E(G)$ is locally finite, we may apply theorem 3.1 to obtain

$$A\,E(G) = B\,E(G) = H^{\mathfrak{s}}K^{\mathfrak{s}}E(G) .$$

Now G/A is Černikov. Hence $E(G) \leq A$.

Similarly, $E(H) \leq H^{\mathfrak{s}}$, $E(K) \leq K^{\mathfrak{s}}$.

By lemma 3.2 we have

$$E(G) = E(H)\,E(K) .$$

Hence $G^{\mathfrak{s}} = H^{\mathfrak{s}}K^{\mathfrak{s}}$.

Corollary Let H be a perfect ascendant subgroup of a group G, and let K be an ascendant subgroup of G. Then if $H, K \in$ Min-sn H permutes with K.

Proof This follows from theorem E and lemma 1.6 .

Theorem F Let $G = \langle H, K \rangle$ $H, K \in$ Min-sn and H, K asc G.

Then $G/G^{L\mathfrak{N}} \in L\mathfrak{N}$

and $G^{L\mathfrak{N}} = \langle H^{L\mathfrak{N}}, K^{L\mathfrak{N}} \rangle = H^{L\mathfrak{N}}K^{L\mathfrak{N}}$.

Proof As in theorem E, we obtain that $G \in$ Min-sn.
Hence $G/G^{L\mathfrak{N}} \in L\mathfrak{N}$.

Locally nilpotent groups satisfying Min-sn are Černikov and soluble (see, for example, [25] p. 154).

Therefore $H/H^{L\mathfrak{N}}$, $K/K^{L\mathfrak{N}}$, and $G/G^{L\mathfrak{N}}$ are soluble.

By theorem E

$$G^{\mathfrak{s}} = H^{\mathfrak{s}}K^{\mathfrak{s}} \leq H^{L\mathfrak{N}}K^{L\mathfrak{N}} .$$

So without loss of generality, we may assume $G^{\mathfrak{s}} = 1$, and G is soluble. But then G is Černikov and so locally finite.

Hence by theorem 3.1

$$G^{L\mathfrak{N}} = H^{L\mathfrak{N}} K^{L\mathfrak{N}} .$$

We note that the equivalent theorem is not true for nilpotent residuals, since the join of two ascendant nilpotent subgroups which satisfy Min-sn need not be nilpotent.

Let $G = \langle t, A ; a^t = a^{-1}, t^2 = 1, a \in A \rangle$ where $A \cong C_{2^\infty}$. Then $T = \langle t \rangle$ and A are ascendant in G and belong to Min-sn $\cap\, \mathfrak{N}$. But G is not nilpotent.

PART TWO

CRITERIA FOR SUBNORMALITY AND ASCENDANCY IN SOLUBLE AND GENERALIZED SOLUBLE GROUPS

Chapter 4 Subnormality and Ascendancy in Soluble and Generalized Soluble Groups.

The results in this chapter are based on those proved by Wielandt for finite groups in [36].

Following Kurosh [16] we use the following definitions:

A class of generalized soluble groups is a class $\mathfrak{X}$ satisfying

$$\mathfrak{F} \cap \bar{\mathfrak{X}} \leq \bar{\mathfrak{S}} \leq \mathfrak{X}$$

i.e. a finite $\mathfrak{X}$-group is soluble and every soluble group is an $\mathfrak{X}$-group.

A group is SN* or SI* if it has respectively an ascending or ascending normal series with abelian factors.

Sec. 1 Soluble Groups.

The main theorems in this section will be:

Theorem G.1 Let G be a soluble group of derived length d, and let H be a subgroup of G. Suppose that for all sequences $h_1, \ldots, h_n$; $h_i \in H$, n a fixed integer, and for all $g \in G$

$$[\, g, h_1, \ldots, h_n \,] \in H$$

Then $H \triangleleft^{nd} G$.

Corollary G.1 Let G be a soluble group of derived length d and H a subgroup of G.

Then if $H \triangleleft^{n} \langle H,g \rangle$ $\forall g \in G$, n a fixed integer, then $H \triangleleft^{nd} G$.

The corollary follows easily from the theorem since if $L = \langle H,g \rangle$, some $g \in G$, by hypothesis we have $\gamma_n L H^n \leq H$.

Hence the subgroup H satisfies also the hypothesis of the theorem.

<u>Theorem H.1</u> Let G be a soluble group and let H be a subgroup of G. Then H is subnormal in G if and only if there exists an integer $n \geq 0$ and for each $g \in G$ a sequence $H_1, \ldots, H_n$ of generating sets of H such that $[g, H_1, \ldots, H_n] \leq H$

For our proof of theorem G.1 we examine closely the following special case:

<u>Lemma 4.1</u> Let G be a group and let H be a subgroup of G, and suppose that $G = HA$ where $A \triangleleft G$ and $A \in \mathfrak{A}$.

If, for all sequences $h_1, \ldots, h_n$; $h_i \in H$, n a fixed integer, and for all $g \in G$

$$[g, h_1, \ldots, h_n] \in H$$

then $H \triangleleft^n G$.

<u>Proof</u> Since $A \triangleleft G$ we have $A \cap H \triangleleft H$, and since A is abelian $H \cap A \triangleleft A$. Since the hypotheses of the theorem remain true on taking homomorphic images of G, we may factor by $H \cap A$ and assume $H \cap A = 1$.

Now let $a \in A$. Let $g_1, \ldots, g_n$ be any sequence in G. Since $G = HA$ we have that $g_i = a_i h_i$ where $a_i \in A$, $h_i \in H$.

Hence $[a, g_1, \ldots, g_n] = [a, a_1 h_1, \ldots, a_n h_n]$

Now $[a, a_1 h_1] = [a, h_1][a, a_1]^{h_1}$.

by simple commutator calculus

$= [a, h_1]$ since $A \in \mathfrak{A}$

Now $A \lhd G$ and so $[a, h_1] \in A$. Hence by a simple induction we have

$$[a, h_1, \ldots, h_n] = [a, g_1, \ldots, g_n].$$

But by hypothesis $[a, h_1, \ldots, h_n] \in H$ and since $A \lhd G$, $[a, h_1, \ldots, h_n] \in A$.

So $[a, h_1, \ldots, h_n] \in A \cap H = 1$.

So $[a, g_1, \ldots, g_n] = 1$ for all sequences $g_1, \ldots, g_n$.

Hence $a \in \zeta_n(G)$ and since a was arbitrary $A \leq \zeta_n(G)$.

Now since

$$H\zeta_i(G) / \zeta_i(G) \lhd H\zeta_{i+1}(G) / \zeta_i(G) \quad \forall i \geq 0$$

with $\zeta_0(G) = 1$

we have

$$H \lhd H\zeta_1(G) \lhd \cdots \lhd H\zeta_n(G) = G$$

i.e. $H \lhd^n G$.

We can now easily prove theorem G.1 with the aid of the lemma.

<u>Proof</u> (of theorem G.1) We use induction on d. If $d = 1$ then G is abelian and H is obviously a normal subgroup. Hence assume $d > 1$ and that the theorem is true for soluble groups of derived length less than d. Let A be the last but one term of the derived series of G. Then $A \lhd G$ and $A \in \mathfrak{A}$.

Since the hypotheses of the theorem remain true on taking homomorphic images of G, we have by induction on d, since G/A has derived length $d-1$, that

$$HA/A \triangleleft^{n(d-1)} G/A$$

i.e. $HA \triangleleft^{n(d-1)} G$.

But the group HA satisfies the hypotheses of lemma 4.1 and so by the lemma

$$H \triangleleft^{n} HA$$

Hence $H \triangleleft^{nd} G$.

We prove theorem H.1 in a fairly similar way using the following lemma in place of lemma 4.1.

<u>Lemma 4.2</u> Let $G = HA$ where $A \triangleleft G$, $A \in \mathfrak{A}$. Then if there exists an integer $n \geq 0$ and for each $g \in G$ a sequence $H_1, \ldots, H_n$ of generating sets of H such that $[g, H_1, \ldots, H_n] \leq H$, then $H \triangleleft^n G$.

<u>Proof</u> As in lemma 4.1 we have $H \cap A \triangleleft G$ and since our hypotheses remain true under homomorphic images of G we may factor by $H \cap A$ and assume $H \cap A = 1$.

Let $a \in A$. Then by hypothesis, for a sequence $H_1, \ldots, H_n$ of generating sets of H we have

$$[a, H_1, \ldots, H_n] \leq H.$$

Since $A \triangleleft G$ we have $[a, H_1, \ldots, H_n] \leq A$ and so $[a, H_1, \ldots, H_n] \leq H \cap A = 1$.

Hence $[a, H_1, \ldots, H_{n-1}]$ centralizes all elements in a generating set of H and so is contained in the centralizer of H.

Let $l_1 \in [a, H_1, \ldots, H_{n-1}]$. Then $l_1 \in A \trianglelefteq G$

$\therefore [l_1, AH] = 1.$

(Let $g \in G$, then $g = a'h$ some $h \in H$, $a' \in A$.

So $[l_1, g] = [l_1, a'h]$

$$= [l_1, h][l_1, a']^h$$

$$= [l_1, h] \quad \text{since } A \in \mathfrak{A}$$

$$= 1 \quad \text{since } l_1 \in C_G(H) \;)$$

Hence $l_1 \in \zeta(G)$ and so $[a, H_1, \ldots, H_{n-1}] \leq \zeta(G)$.

Now suppose that $[a, H_1, \ldots, H_{n-i}] \leq \zeta_i(G)$ and let $l_{i+1} \in [a, H_1, \ldots, H_{n-(i+1)}]$.

As before if $g = ha'$ some $h \in H$, $a' \in A$

$$[l_{i+1}, g] = [l_{i+1}, h]$$

Since we supposed $[a, H_1, \ldots, H_{n-i}] \leq \zeta_i(G)$ we have $[l_{i+1}, g] \in \zeta_i(G) \;\; \forall g \in G$.

Hence $l_{i+1} \in \zeta_{i+1}(G)$ and $[a, H_1, \ldots, H_{n-(i+1)}] \leq \zeta_{i+1}(G$

By induction on i we obtain $A \leq \zeta_n(G)$.

As before, since

$$H\zeta_i(G)/\zeta_i(G) \quad \triangleleft \quad H\zeta_{i+1}(G)/\zeta_i(G)$$

with $\zeta_0(G) = 1$, we have

$$H \triangleleft H\zeta_1(G) \triangleleft \ldots \triangleleft H\zeta_n(G) = G$$

Hence $H \triangleleft^n G$.

<u>Proof of Theorem H.1</u> To show sufficiency of the condition we use induction on d, the derived length of G. If $d = 1$ then G is abelian and $H \triangleleft G$.

So we may assume $d > 1$ and that the theorem is true for all soluble groups of derived length less than d. We in fact prove by induction that $H \triangleleft^{nd} G$.

Let A be the last but one term of the derived series of G. Then $A \in \mathfrak{A}$ and $A \triangleleft G$. Since the hypotheses of the theorem remain true on taking quotients of G we have by induction on d

$$AH/A \triangleleft^{n(d-1)} G/A$$

Hence $HA \triangleleft^{n(d-1)} G$.

Since the group HA satisfies the conditions of lemma 4.2 we obtain $H \triangleleft^n HA$ and hence $H \triangleleft^{nd} G$.

To show the necessity of the condition let H sn G and have normal closure series

$$H = L_n \triangleleft L_{n-1} \triangleleft \ldots \triangleleft L_0 = G$$

Then if $H_1, \ldots, H_m$ are generating sets of H and $g \in G$ we have

$$[g, H_1, \ldots, H_m] \leq L_m \qquad 0 \leq m \leq n.$$

<u>Sec. 2</u> <u>Generalized Soluble Groups</u>

For SI^* - groups we obtain the following two theorems analogous to theorems G.1 and H.1 in section 1 of this chapter.

Theorem G.2 Let G be an SI^* - group, and let H be a subgroup of G. Suppose that for each $g \in G$ and for all sequences $h_1, \ldots, h_i, \ldots$ of H that there exists an integer $n = n(g)$ such that $[g, h_1, \ldots, h_n] \in H$. Then H is ascendant in G.

Corollary G.2 Let G be an SI^* - group and let H be a subgroup of G. Then H asc $\langle H, g \rangle$ $\forall g \in G$ implies that H asc G.

Theorem H.2 Let G be an SI^* - group and let H be a subgroup of G. Then if for each $g \in G$ there exists an integer $n = n(g)$ and a sequence $H_1, \ldots, H_n$ of generating sets of H such that $[g, H_1, \ldots, H_n] \leq H$ then H asc G.

We mention that as shown in [10] the condition stated in theorem H.2 is not necessary for ascendancy, for let G be the wreath product $C \wr B$ of a group C of type C_{2^∞} by an elementary abelian group of order 4 generated by i and j. Then G is locally nilpotent and satisfies the minimal condition, and hence is hypercentral. Let $H = \langle C, i \rangle$, then H asc G since G is hypercentral. Now for any sequence $H_1, \ldots, H_n$ of generating sets of H we have that

$$[\,[\,j, c\,]_{n-1}\, i\,] \in [\,j, H_1, \ldots, H_n\,] \quad \text{where}$$

c is any generator of C.

By a simple induction we can show that

$$[\,[\,j, c\,]_{n-1}\, i\,] = \left(c^{-2^{n-2}}\, c^{2^{n-2} j}\, c^{2^{n-2} i}\, c^{-2^{n-2} ij} \right)^{(-1)^{n-2}}$$

But if this element lies in H we must have $c^{2^{n-2}} = 1$. However for each n we can always find a generator c_1 of C such that $c_1^{2^{n-2}} \neq 1$. Hence we can not find a sequence

$H_1, \ldots, H_n$ of generating sets of H such that $[\, j, H_1, \ldots, H_n \,] \leq H$

We note, however, that if H is finitely generated then the condition is necessary [See [10] lemma 2.4].

We begin the proof of theorem G.2 with the following lemma.

<u>Lemma 4.3</u> Let G be a group and let H be a subgroup of G. Suppose that $G = HA$ where $A \lhd G$ and $A \in \mathfrak{A}$ and that for each $g \in G$ and for all sequences $h_1, \ldots, h_i, \ldots$ of H there exists an integer $n = n(g)$ such that $[\, g, h_1, \ldots, h_n \,] \in H$ Then $H \text{ asc } G$.

<u>Proof</u> We have $H \cap A \lhd H$ since $A \lhd G$ and $H \cap A \lhd A$ since $A \in \mathfrak{A}$. Hence $H \cap A \lhd G$ and since the hypotheses of the lemma remain true on taking homomorphic images of G, we may factor by $H \cap A$ and assume $H \cap A = 1$.

Now $C_A(H)$ is normalized by H and $C_A(H) \lhd A$ since $A \in \mathfrak{A}$ and so $C_A(H) \lhd G$.

Hence we can define groups A_α for ordinals α by

$$A_0 = 1 \qquad A_{\alpha+1}/A_\alpha = C_{A/A_\alpha}(H)$$

and $A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha$ for limit ordinals λ.

Then $A_\alpha H \lhd A_{\alpha+1} H$ and so if $\bar{A} = \bigcup_\alpha A_\alpha$ we have $H \text{ asc } \bar{A} H$.

Since our hypotheses remain true on taking homomorphic images of G, it is enough to show that if $A > 1$ then $C_A(H) > 1$ so that $A = \bar{A}$ and $H \text{ asc } AH = G$.

Hence, suppose for a contradiction that $A \neq 1$ but $C_A(H) = 1$. Then if $1 \neq a_1 \in A$ then $a_1 \notin C_A(H)$. Hence there exists

$h_1 \in H$ such that

$$[\, a_1, h_1 \,] \neq 1$$

Let $a_2 = [\, a_1, h_1 \,]$. Then $a_2 \in A$ since $A \lhd G$ and $a_2 \neq 1$. So again $a_2 \notin C_A(H)$ and so there exists $h_2 \in H$ such that

$$[\, a_2, h_2 \,] \neq 1.$$

Let $a_3 = [\, a_2, h_2 \,]$. Continuing in this way and defining $a_n = [\, a_{n-1}, h_{n-1} \,]$ we can find a sequence $h_1, h_2, \ldots, h_n, \ldots$ such that $a_n \neq 1$ for any n. But this contradicts the hypotheses of the lemma.

Hence if $A \neq 1$, $C_A(H) > 1$ and so $A = \bar{A}$ and H asc G.

<u>Corollary</u> Let $G = HA$ where H is a subgroup of G and A a normal subgroup, $A \in \mathfrak{A}$. Then if H asc $\langle H, g \rangle$ $\forall g \in G$ we have H asc G.

<u>Proof</u> This follows from lemma 4.3 and the following lemma.

<u>Lemma 4.4</u> Let G be a group and let H be an ascendant subgroup of G.

For any two sequences

$$g_1, g_2, \ldots \qquad h_1, h_2, \ldots \qquad g_i \in G,\ h_i \in H$$

such that

$$g_{i+1} = [\, g_i, h_i \,]$$

there is an $m > 0$ such that $g_m \in H$.

<u>Proof</u> Let $g_1, g_2, \ldots$ $h_1, h_2, \ldots$ be two sequences with $g_i \in G$, $h_i \in H$ such that $g_{i+1} = [\, g_i, h_i \,]$. Now H asc G so there exists an ascending series

$$H = H_0 < H_1 < H_2 \dots \quad H_\rho = G.$$

Suppose $g_i \notin H$ for all i.

Then $g_i \in H_{\alpha_i+1} \setminus H_{\alpha_i}$ for some ordinal α_i.

Now since $H_{\alpha_i} < H_{\alpha_{i}+1}$ we have

$$g_{i+1} = [g_i, h_i] \leq [H_{\alpha_i+1}, H] \leq H^{H_{\alpha_i+1}} \leq H_{\alpha_i}$$

So $\alpha_{i+1} < \alpha_i$.

Hence we can find an infinite descending chain of ordinals $\alpha_1 > \alpha_2 > \dots$ which is impossible.

Hence $g_m \in H$ for some m.

Theorem G.2 now follows easily from lemma 4.3.

<u>Proof of Theorem G.2</u> Let $1 = G_0 \lhd G_1 \lhd \dots G_\rho = G$ be an ascending abelian series with $G_\alpha \lhd G \ \forall \alpha \leq \rho$.

Now $HG_{\alpha+1}/G_\alpha = HG_\alpha/G_\alpha \cdot G_{\alpha+1}/G_\alpha$

and $G_{\alpha+1}/G_\alpha \in \mathfrak{A}$ $\quad G_{\alpha+1}/G_\alpha \lhd HG_{\alpha+1}/G_\alpha$

Since the group $HG_{\alpha+1}/G_\alpha$ satisfies the hypotheses of lemma 4.3 we may apply this lemma to obtain

$$HG_\alpha/G_\alpha \text{ asc } HG_{\alpha+1}/G_\alpha$$

Hence HG_α asc $HG_{\alpha+1}$. This is true for all $\alpha < \rho$ so

$$H = G_0 H \ \text{asc} \ G_\rho H = G, \text{ i.e. } H \text{ asc } G.$$

<u>Proof of Corollary G.2</u> This follows from theorem G.2 and lemma 4.4.

We now prove theorem H.2 in a similar way with the following lemma playing the role of lemma 4.3.

<u>Lemma 4.5</u> Let G be a group and H a subgroup of G. Let $G = HA$ where $A \triangleleft G$, $A \in \mathfrak{A}$. Then if for each $g \in G$ there exists $n = n(g) \geq 0$ and a sequence $H_1, H_2, \ldots, H_n$ of generating sets of H such that $[g, H_1, \ldots, H_n] \leq H$ then H asc G.

<u>Proof</u> As in lemma 4.3 we may assume $H \cap A = 1$. As before $C_A(H) \triangleleft G$ and we define groups A_α for ordinals α by $A_0 = 1$ $\quad A_{\alpha+1}/A_\alpha = C_{A/A_\alpha}(H)$

$$A_\lambda = \bigcup_{\alpha<\lambda} A_\alpha \quad \text{for limit ordinals } \lambda.$$

Then $A_\alpha H \triangleleft A_{\alpha+1} H$ and so if $\bar{A} = \bigcup A_\alpha$ we have H asc $\bar{A} H$.

As in lemma 4.3 it is enough to show that if $A > 1$ then $C_A(H) > 1$.

Suppose for a contradiction that $A \neq 1$ and $C_A(H) = 1$. Let $1 \neq a \in A$, then $a \notin C_A(H)$. By hypothesis there exists a sequence of generating sets $H_1, \ldots, H_m$ of H, $m = m(a)$ such that

$$[a, H_1, \ldots, H_m] \leq H.$$

But since $A \lhd G$ we have

$$[a, H_1, \ldots, H_m] \leq H \cap A = 1.$$

Now $a \notin C_A(H)$ so there exists $h_1 \in H_1$ such that

$[a, h_1] \neq 1$. Since $[a, h_1] \in A$ we know

$[a, h_1] \notin C_A(H)$.

Hence there exists $h_2 \in H_2$ such that

$$[a, h_1, h_2] \neq 1.$$

So continuing in this way we obtain a sequence $h_1, h_2, \ldots, h_m$; $h_i \in H_i$ such that

$$[a, h_1, h_2, \ldots, h_m] \neq 1.$$

But $[a, h_1, h_2, \ldots, h_m] \in [a, H_1, \ldots, H_m] = 1$.

This contradiction shows that $C_A(H) > 1$ if $A > 1$ and hence H asc G.

Proof of Theorem H.2 Let $1 = G_0 \lhd G_1 \lhd \ldots \quad G_\rho = G$ be an ascending abelian series with $G_\alpha \lhd G \;\; \forall \alpha \leq \rho$.

Applying lemma 4.5 to the group $HG_{\alpha+1}/G_\alpha$ we obtain in a similar way to the proof of theorem G.2, that HG_α asc $HG_{\alpha+1}$ and hence $H = G_0H$ asc $G_\rho H = G$, i.e. H asc G.

SN* - groups. We are able to obtain similar theorems to theorems G.2 and H.2 with the use of lemmas 4.3 and 4.5, provided that we can obtain an ascending abelian series of G, an SN* - group, which is normalized by the subgroup H.

This is possible by putting conditions on our subgroup H. We do not know whether the theorems remain true for arbitrary subgroups H.

We introduce a subclass $\mathfrak{F}^*$ of the class $\mathfrak{F}$ of finitely generated groups by $G \in \mathfrak{F}^*$ if for all $x \in G$, $\langle x^G \rangle$ is finitely generated.

We denote by Max-asc the class of all groups which have the maximal condition on the set of all ascendant subgroups.

<u>Theorem G.3</u> Let G be an SN^* - group, and let H be a subgroup of G. Suppose that for each $g \in G$ and for all sequences $h_1, \ldots, h_i, \ldots$ of H that there exists an integer $n = n(g)$ such that $[g, h_1, \ldots, h_n] \in H$. Then if $H \in \mathfrak{F}^* \cup (\mathfrak{F} \cap \text{Max-asc})$ then H asc G.

<u>Corollary G.3</u> Let G be an SN^* - group and H a subgroup of G. Suppose that H asc $\langle H, g \rangle$ $\forall g \in G$. Then if $H \in \mathfrak{F}^* \cap (\mathfrak{F} \cap \text{Max-asc})$ then H asc G.

<u>Theorem H.3</u> Let G be an SN^* - group and let H be a subgroup of G. Then if for each $g \in G$ there exists an integer $n = n(g)$ and a sequence $H_1, \ldots, H_n$ of generating sets of H such that $[g, H_1, \ldots, H_n] \leq H$ then if $H \in \mathfrak{F}^* \cup (\mathfrak{F} \cap \text{Max-asc})$, H asc G.

Let G be an SN^* - group and let

$1 = G_0 \triangleleft \cdots \quad G_\rho = G$ be an ascending series of G with abelian factors.

Let $\overline{G}_\alpha = \bigcap_{h \in H} G_\alpha^h$

Then by Corollary 2.2 of [12], since $H \in \mathfrak{X}^* \cup (\mathfrak{X} \cap \text{Max-asc})$ then

$$\overline{G}_\alpha \lhd \overline{G}_{\alpha+1} \quad \forall\, \alpha < \rho \quad \text{and} \quad \overline{G}_\lambda = \bigcup_{\beta<\lambda} \overline{G}_\beta \quad \lambda\text{, a limit}$$

ordinal. Also if α is not a limit ordinal then

$$\overline{G}_\alpha / \overline{G}_{\alpha-1} \in \mathfrak{A} .$$

Hence $\{ \overline{G}_\alpha \}_{\alpha \leq \rho}$ is an ascending abelian series of G with all terms normalized by H.

Hence for all $\alpha < \rho$

$$\overline{G}_\alpha \lhd H \overline{G}_{\alpha+1} .$$

Using this fact we are able to apply lemmas 4.3 and 4.5 and the proofs of theorems G.3 and H.3 follow the same lines as theorems G.2 and H.2 respectively.

The proof of corollary G.3 is the same as the proof of corollary G.2.

Locally Soluble Groups.

Let G be a locally soluble group and let H be a subgroup of G. Then if $H \text{ sn } \langle H, g \rangle \ \forall\, g \in G$, we need not have $H \text{ asc } G$, as the following example shows.

Example. Let ρ denote the first uncountable ordinal and for each $\alpha < \rho$ let H_α be cyclic with prime order p.

As ascending chain of groups $\{ W_\alpha : \alpha \leq \rho \}$ is defined by the rules

$$W_0 = 1 \qquad W_{\alpha+1} = H_{\alpha+1} \wr W_\alpha \quad \text{and} \quad W_\lambda = \bigcup_{\beta<\lambda} W_\beta$$

for each ordinal $\alpha < \rho$ and each limit ordinal $\lambda \leq \rho$. Here, the wreath product is the standard one and W_α is embedded in $W_{\alpha+1}$ in the natural way.

The group W is a locally finite p - group. So let H be a finitely generated subgroup. Then $< H, g >$ is also finitely generated for each $g \in W$, and so is a finite p - group and hence nilpotent.

So $H \text{ sn } < H, g > \ \forall g \in W$.

But H is not ascendant in W, since the Gruenberg radical of W is trivial. [See Robinson [26] p. 28 Theorem 6.27].

Chapter 5 Subnormality in Soluble Groups with Finiteness Conditions.

Sec. 1 Introduction

In [20] Peng examines the subnormality of a subgroup H in a soluble - by - Max group where H satisfies the condition that there exists an integer $n \geq 0$ such that $[g,{}_n h] \in H$ $\forall g \in G$ and $\forall h \in H$.

He shows that this implies H is subnormal in G, provided that H is polycyclic - by - finite.

This theorem is not true for arbitrary subgroups H even when G is a metabelian group, for example let G be the standard wreath product of a group of order 2 with a countably infinite elementary abelian 2-group, H. Then $[x,y,y,y] = 1$ for all $x, y \in G$. Although every k-generator subgroup of G is nilpotent of class $k+1$ for $k \geq 2$, G is itself not nilpotent. (A proof of these is given in [19] p. 98.)

So $[g,{}_3 h] \in H$ $\forall g \in G$ and $h \in H$. But H is shown in [20] to be its own normalizer in G.

We note that our conditions in theorem G.1 were stronger than those considered by Peng, but that theorem G.1 was true for arbitrary subgroups.

In [20], the case is also considered when n is not a fixed integer, but depends on g and h. Provided that H is polycyclic - by - finite, H is shown to be ascendant in G.

We shall be concerned here with a weaker condition that implies subnormality in soluble groups; first by imposing finiteness conditions on the group G, and then considering an arbitrary soluble group but imposing conditions on the subgroup H.

Sec. 2 The Finite Case.

We would like to thank Dr. Brian Hartley for suggesting the following result, and for indicating the method of proof. We do not know whether the theorem is true for non-soluble finite groups, and we leave this as an open question.

Theorem 5.1 Let G be a finite soluble group. Let $H = \langle h_i \mid 1 \leq i \leq m \rangle$. Then H is subnormal in G, if and only if $[g,_n h_i] \in H \ \forall g \in G$, $n = |G|$, $i = 1, \ldots, m$.

To prove this we introduce two concepts:

Let G be any group and let $a \in G$. We define subgroups $X_r(a)$ for integers $r \geq 0$ by

$$X_r(a) = \langle a, [g,_r a] ; g \in G \rangle$$

Then $X_r(a) \geq X_{r+1}(a)$ for all integers $r \geq 0$.

Following Wielandt, we define the subnormalizer of a subgroup X of a group G, to be the intersection of all subnormal subgroups of G containing X.

We denote the subnormalizer of X by X''^G. In general X''^G need not be subnormal in G. For let G be the infinite dihedral group:

$$D_\infty = \langle a, x \mid a^{-1} x a = x^{-1}, a^2 = 1 \rangle \quad \text{and } X = \langle a \rangle .$$

If $N = \langle x \rangle$ then $a^{-1} n a = n^{-1} \ \forall n \in N$ and so

$$X_r(a) = \langle x^{2^r}, a \rangle .$$

Then $X_r(a) \lhd X_{r-1}(a)$ and so $X_r(a)$ sn $G \ \forall r$.

But if X''^G is subnormal in G then $X_m(a) \leq X''^G$ for some integer m. Therefore the chain

$$\ldots \leq X_i(a) \leq X_{i-1}(a) \leq \ldots \leq \langle a^G \rangle \leq G$$

terminates after finitely many steps.

But for all positive integers r

$$X_r(a) < X_{r-1}(a)$$

Hence X''^G is not subnormal in G.

We note however, that when G satisfies the minimal condition on subnormal subgroups, X''^G is a subnormal subgroup of G, since the intersection of finitely many subnormal subgroups is subnormal.

Theorem 5.1 will follow from:

Lemma 5.1 Let G be a finite soluble group and let $a \in G$. Then if $n = |G|$ we have

$$< a >''^G = X_r(a), \quad \forall r > n.$$

Proof Since G is finite we have $< a >''^G$ is subnormal in G, and since $|G| = n$ the subnormal index is bounded by n. Hence

$$X_r(a) \leq X_n(a) \leq < a >''^G$$

So it is enough to show that $X_r(a)$ is subnormal in G.

Suppose for a contradiction that this is false and let G be a minimal counterexample.

Let $X = X_r(a)$ where $r > n = |G|$. Suppose X is not subnormal in G.

Let $G_1 = < a^G >$. If $G_1 < G$ then the subgroup

$$X_1 = < a, [g_1, {}_m a] ; g_1 \in G_1 > \quad m = |G_1|$$

is subnormal in G_1. Since $G_1 \lhd G$ we have X_1 is subnormal in G.

Hence $< a >''^G \leq X_1$ and so $X \leq X_1$. Since $m = |G_1|$ we have

$$< a, [g_1, {}_r a] ; g_1 \in G_1 > = X_1 \text{ and so } X_1 \leq X.$$

Hence $X = X_1$, and X is subnormal in G which is a contradiction. Hence $G = < a^G >$.

Let N be a minimal normal subgroup of G.

Then XN/N sn G/N .

Hence $XN = G$ since $G = \langle a^G \rangle$.

Since G is soluble, N is an elementary abelian p-group, for some prime p. Hence $X \cap N \triangleleft G$. If $X \cap N = N$ then $N \leq X$ and $G = X$. So $X \cap N = 1$. Let Y be a proper maximal subgroup of G containing X.

Then $Y = Y \cap XN = X(Y \cap N)$. Since $Y \cap N \triangleleft G$, $Y \cap N = 1$ since $Y \neq G$. Hence $Y = X$ and X is a maximal subgroup of G.

Since X is not subnormal in G, and X is a maximal subgroup we have $X = N_G(X)$.

Now $C_G(a) \leq N_G(X) \leq X$.

Hence $C_N(a) = 1$.

Let N_1 be a minimal a - invariant subgroup of N.

Then $[N_1, a] = N_1$

$$= [N_1, a, \ldots, a]$$

$$= [N_1, {}_r a] \leq X$$

But this is a contradiction to the definition of X.

Hence X is subnormal in G and the lemma is proved.

<u>Proof of theorem 5.1</u>

We have $H = \langle h_i \mid 1 \leq i \leq m \rangle$

$= \langle X_n(h_i) \,;\, i = 1, \ldots, m \rangle$ where $n = |G|$.

By lemma 5.2 we know that each $X_n(h_i)$ is subnormal in G, $1 \leq i \leq m$.

Since G is finite, the join of the finitely many subgroups $X_n(h_i)$ is subnormal.

Hence H is subnormal in G. Obviously the converse holds.

Sec. 3 Subnormality in Polycyclic Groups.

We begin with a result proved by Kegel in [15]. As our proofs in this section will depend heavily on this result, we give a proof here and note that it is an alternative one to that given in [15].

Theorem 5.2 (Kegel) Let G be a polycyclic group and X a subgroup of G. Suppose that X is subnormal in G modulo normal subgroups of finite index in G, then X is subnormal in G.

We shall need the following lemmas in the proof.

Lemma 5.2 Let G be a polycyclic group and X a subgroup of G. Let X sn G modulo normal subgroups of finite index of G. Then if $X \leq H < G$ we have that X is subnormal in H modulo normal subgroups of finite index of H.

Proof Let $X \leq H < G$ and let M be a normal subgroup of finite index of H. Then using Mal'cevs theorem given in theorem 2.1 we have that there exists a subgroup F of G with finite index in G, such that $M = H \cap F$.

Let $F_G = \mathrm{Core}_G(F)$. Then by hypothesis, there exists an integer r such that

$$
\begin{aligned}
& [\, G, {}_r X \,] \leq XF_G \\
\text{i.e.}\quad & [\, H, {}_r X \,] \leq XF_G \cap H \\
& \qquad\qquad = X(\, H \cap F_G) \\
& \qquad\qquad \leq XM\,.
\end{aligned}
$$

Hence X is subnormal in H modulo M.

Lemma 5.3 Let M be a non-trivial normal subgroup lying in the hypercentre of a group G, then $M \cap \zeta(G) \neq 1$.

Proof See Robinson [25] p. 47 Lemma 2.16.

Proof of Theorem 5.2 We use induction on d, the derived length of G. Since the hypothesis of the theorem remains true on taking homomorphic images of G, if N is the last but one term of the derived series of G we obtain by induction

$$XN/N \text{ sn } G/N$$

Hence XN sn G.

By lemma 5.2 the group XN also satisfies the hypothesis of the theorem, so we may assume $G = XN$.

Now $X \cap N \triangleleft N$ since $N \in \mathfrak{A}$, and $X \cap N \triangleleft X$ since $N \triangleleft G$. So $X \cap N \triangleleft G$, and we may assume without loss of generality that $X \cap N = 1$.

Case 1 N is finite.

If X is finite we have by hypothesis that X is subnormal in G. Hence we assume X is infinite.

Since N is finite $|G:C_G(N)| < \infty$. Now $C_G(N) \triangleleft G$ since $N \triangleleft G$. Hence $X \cap C_G(N) \triangleleft X$ and $X \cap C_G(N)$ is normalized by N.

So $X \cap C_G(N) \triangleleft G$ and so we may factor by $X \cap C_G(N)$ and assume $X \cap C_G(N) = 1$.

So $X \cong X/X \cap C_G(N) \cong XC_G(N)/C_G(N) \in \mathfrak{F}$

i.e. X is finite and so X sn G.

Case 2 N is infinite.

Let N_o be the periodic subgroup of N. Then N_o is finite and by case 1 we have X sn XN_o. Since N_o char $N \triangleleft G$, $N_o \triangleleft G$.

Hence we may assume, that $N_o = 1$.

We now have that N is free abelian of finite rank.

Now N^p char $N \triangleleft G \;=\; N^p \triangleleft G$ and N/N^p is finite.

By case 1

$$X N^p / N^p \quad \text{sn} \quad G/N^p$$

and this is true for all primes p. Hence $[\, G, {}_n X \,] \leq X N^p$ for some integer $n = n(p)$.

So $[\, N, {}_n X \,] \leq X N^p \cap N$

$$= N^p (X \cap N)$$

$$= N^p \quad \text{since} \quad X \cap N = 1.$$

As $N \in \mathfrak{A}$ and $N \triangleleft G$

$$[\, N, {}_n G \,] = [\, N, {}_n X \,]$$

Hence $N/N^p \leq \zeta_n (G/N^p)$

Let G have rank l. Then N/N^p has order dividing p^l.
By lemma 5.3 we have

$$N/N^p \leq \zeta_l (G/N^p) \quad \text{and so} \quad [\, N, {}_l X] \leq N^p .$$

Since l does not depend on the prime p we obtain

$[\, N, {}_l X \,] \leq N^p$ for all primes p,

i.e. $[\, N, {}_l X \,] \leq \bigcap_p N^p$

But N is free - abelian (of finite rank) and so $\bigcap_p N^p = 1$.

Hence $[\, N, {}_l G \,] = [\, N, {}_l X \,] = 1$ and
$N \leq \zeta_l(G)$.

Since $X \zeta_i(G) / \zeta_i(G) \quad \triangleleft \quad X \zeta_{i+1}(G) / \zeta_i(G) \qquad \forall i$

where $\zeta_o(G) = 1$ we have

$X \triangleleft X \zeta_1(G) \triangleleft \ldots \triangleleft X \zeta_l(G) = G$
i.e. X sn G.

We can now easily prove

Theorem 5.3 Let G be a polycyclic group. Let $H \leq G$ and suppose that $H = \langle h_i \mid 1 \leq i \leq m \rangle$. Then if there exists an integer $n = n(g)$ such that $[g, {}_n h_i] \in H \ \forall g \in G$ $i = 1, \ldots, m$ then H is subnormal in G.

Proof Since the hypotheses of the theorem remain true on taking homomorphic images of G we know, by theorem 5.1, that H is subnormal in G modulo normal subgroups of finite index.

Hence by theorem 5.2, H is subnormal in G.

As was shown before, in the case of the infinite dihedral group the chain

$$\ldots \leq X_i(a) \leq X_{i-1}(a) \leq \ldots \leq \langle a^G \rangle \leq G$$

need not terminate after finitely many steps.

We do not know in general whether, in a polycyclic group G, $a \in G$, we have $X_n(a)$ is subnormal in G for some integer n. However we do obtain

Lemma 5.4 Let G be a polycyclic group and let $a \in G$. Suppose for some integer r that

$$X_r(a) = X_{r+i}(a) \quad \forall i \geq 0$$

then $X_r(a)$ is subnormal in G.

Proof By lemma 5.1 if $X = X_r(a)$ then X is subnormal in G modulo normal subgroups of finite index in G. Hence by theorem 5.2 X is subnormal in G.

Also, when G is a finitely generated abelian - by - finite soluble group, there exists an integer N such that $X_n(a)$ is subnormal in G $\forall n \geq N$. This will be proved in the next section.

Sec. 4 Our main result here is

Theorem I Let G be a soluble group.
Let $H = \langle h_i \mid 1 \leq i \leq m \rangle \leq G$ such that $[g, {}_n h_i] \in H$ $\forall g \in G$, $i = 1, \ldots, m$, n a fixed integer.

Then if H is Min - by - Nilpotent, H is subnormal in G.

For the proof of the theorem, we begin with the following lemmas.

Lemma 5.6 Let G be any group. Let $X \leq G$ and let N be a normal abelian subgroup of G. Let $a \in X$ and suppose that

$$\overline{X} = \langle a, [g_1, {}_r a] \mid g_1 \in XN \rangle$$

and $XN = \overline{X} N$.

Then $\overline{X} \lhd^r XN$.

Proof Without loss of generality we may suppose that $G = XN$. Now $\overline{X} \cap N \lhd \overline{X}$ and $\overline{X} \cap N \lhd N$ since $N \in \mathfrak{A}$. Hence we may factor G by $\overline{X} \cap N$ and assume $\overline{X} \cap N = 1$.

Let $N_1 = N_N(\overline{X})$. Then N_1 is centralized by $\overline{X}$ and $N_1 \lhd N$ since $N \in \mathfrak{A}$. Hence $N_1 \lhd G$.

By induction define N_i for $i > 1$ by

$$N_i / N_{i-1} = N_{N/N_{i-1}} (\overline{X} N_{i-1} / N_{i-1})$$

We wish to show $N = N_r$.

Since $C_G(a) \leq N_G(\overline{X})$ we have

$$C_N(a) \leq N_1$$

and $[n, {}_r a] \in N \cap \overline{X} = 1$.

Hence $[n, {}_{r-1} a] \in C_N(a) \leq N_1$.

Factoring G by N_1 we obtain similarly that

$$[\, n, {}_{r-2}a \,] \in N_2 .$$

By a simple induction we obtain $N = N_r$.

Then $\overline{X} \lhd \overline{X} N_1 \lhd \ldots \lhd \overline{X} N_r = G$

i.e. $\overline{X} \lhd^r G$.

Lemma 5.7 Let G be a soluble group and let $a \in G$. Suppose that for some integer m that

$$X_m(a) = X_{m+r}(a) \quad \forall\, r \geq 0 .$$

Then $X_m(a)$ is subnormal in G.

Proof We use induction on d, the derived length of G. If $d = 1$ then G is abelian and the lemma is trivially true. Hence assume $d > 1$ and let N be the last but one term of the derived series of G.

Then if $X = X_m(a)$ we obtain by induction that

$$XN/N \text{ sn } G/N .$$

Hence XN sn G, and suppose $XN \lhd^l G$.

Let $\overline{X} = \langle a, [\, xn, {}_m a \,] ; xn \in XN \rangle$.

Then $X_{l+m}(a) \leq \overline{X}$.

By hypothesis $X = X_{l+m}(a)$.

Hence $X \leq \overline{X} \leq X$. So $X = \overline{X}$. Hence $XN = \overline{X}N$ and by lemma 5.6 $X = \overline{X}$ sn XN .

Therefore X sn G.

Proof of theorem I.

$H = \langle h_i \mid 1 \leq i \leq m \rangle$. Then by hypothesis $X_n(h_i) \leq H$; $1 \leq i \leq m$.

Since H is Min - by - nilpotent, for each $h_i \in H$ we can find an integer $k \geq n$ such that

$$< h_i, [g, {}_k h_i] > \; = \; < h_i, [g, {}_{k+r} h_i] > \quad \forall r \geq 0 .$$

Let t be the largest such k.

Then $\forall i;\ 1 \leq i \leq m$

$$X_{t+r}(h_i) = X_t(h_i) \quad \forall r \geq 0 .$$

By lemma 5.7 $X_t(h_i)$ is subnormal in G. Now $H \in$ Min - by - Max. Hence, for example by [5] the join H of the finitely many subnormal subgroups $X_t(h_i)$ is subnormal in G.

We finally prove, as a special case

<u>Lemma 5.8</u> Let G be a finitely generated abelian - by - finite soluble group. Let $a \in G$. Then there exists a positive integer M such that $X_r(a)$ sn G $\forall r \geq M$.

<u>Proof</u> Since G is abelian - by - finite there exists an abelian normal subgroup, N, of G with finite index in G.

Let $|G : N| = n$.

Then by lemma 5.1

$$X_n(a)N = X_r(a)N \lhd^n G \quad \forall r \geq n \quad (1)$$

We choose a fixed integer $r \geq n$ and let $X = X_r(a)$.

Then XN sn G.

Let $\overline{X_m(a)} = < a, [xn, {}_m a] ; xn \in XN >$

Now $X \cap N \lhd XN$ and $X/X \cap N \cong XN/N \in \mathfrak{F}$.

Hence there exists an integer, l, such that

$$\overline{X_l(a)}(X \cap N) = \overline{X_s(a)}(X \cap N) \quad \forall s \geq l$$

and $\overline{X_l(a)}(X \cap N)/X \cap N$ is finite.

Hence by lemma 5.7 $\overline{X_1(a)}\,(X \cap N)/X \cap N$ sn $XN/X \cap N$

i.e. $\overline{X_s(a)}\,(X \cap N) = \overline{X_1(a)}\,(X \cap N)$ sn XN .

But XN sn G and so $\overline{X_1(a)}\,(X \cap N)$ sn G .

Hence there exists an integer ρ such that

$X_\rho(a) \leq \overline{X_1(a)}\,(X \cap N) = \overline{X_s(a)}\,(X \cap N) \quad \forall s \geq 1$.

If $\rho \leq 1$ then $X_1(a) \leq \overline{X_1(a)}\,(X \cap N)$.

If $\rho > 1$ then $X_\rho(a) \leq \overline{X_\rho(a)}\,(X \cap N)$.

Hence in either case we obtain the existence of an integer $s \geq r$ such that

$$X_s(a) \leq \overline{X_s(a)}\,(X \cap N) \leq X.$$

But from (1) $X_s(a)N = XN$.

$$\begin{aligned} \text{Hence} \quad X = (XN) \cap X &= (X_s(a)N) \cap X \\ &= X_s(a)(N \cap X) \\ &\leq \overline{X_s(a)}\,(X \cap N) \leq X \end{aligned}$$

i.e. $X = \overline{X_s(a)}\,(X \cap N)$.

Since $\overline{X_s(a)}\,(X \cap N)$ is subnormal in G we have that X is subnormal in G.

Sec. 5 Soluble Minimax Groups.

A group G has finite rank r if every finitely generated subgroup of G can be generated by r elements and r is the smallest integer with this property.

Following Robinson [26], we shall use the term abelian S_1 - group to describe an abelian group of finite rank whose torsion subgroup is Cernikov.

A soluble group is said to be an S_1 - group if the factors in a normal abelian series are abelian S_1 - groups.

A minimax group is a group G which has a series of finite length, each of whose factors satisfies Max or Min.

Since the class of groups of finite rank is P - closed [See Robinson [25] lemma 1.44] and since every polycyclic group and every Cernikov group has finite rank, we see that a soluble minimax group has finite rank. Since a periodic polycyclic group is finite, a periodic soluble minimax group is a Cernikov group. Hence a soluble minimax group is an S_1 - group.

We state the main result of this section.

Theorem J Let G be a soluble minimax group and let $H = < h_i \mid 1 \leq i \leq m > \leq G$ be such that $[g, {}_n h_i] \in H$ $\forall g \in G$, some integer n, $i = 1,...,m$. Then H sn G.

We note that when $H \leq G$ satisfies $[g, {}_n h] \in H$ $\forall g \in G$, $\forall h \in H$ when n is a fixed integer or varies with g and h, then Peng has proved the following theorems in [20] :

Theorem (Peng) Let G be a (soluble - minimax) - by - Min group and let H be any subgroup of G. If there exists an integer $n \geq 0$ such that $[g, {}_n h] \in H$ $\forall g \in G$ and $h \in H$ then H sn G.

Theorem (Peng) Let G be a (soluble of finite rank) - by - Min group and let H be any subgroup of G. If, for each $g \in G$ and $h \in H$ there exists an integer $n = n(g, h) \geq 0$ such that $[g, {}_n h] \in H$ then H asc G.

We begin by defining:

A group is said to be radicable, if each element is an n - th power, for every positive integer n.

Lemma 5.9 Let A be a normal radicable abelian subgroup of a group G, and let H be a subgroup of G such that

$$[\underbrace{A, H, \ldots, H}_{r}] = 1$$

for some positive integer r.

If H/H' is periodic then $[A, H] = 1$.

Proof See Robinson [25] lemma 3.13 .

Lemma 5.10 Let G be a soluble group and suppose $G = AH$ where $A \triangleleft G$, $A \in \mathfrak{A}$.

Then if $H = \langle h_i \mid 1 \leq i \leq m \rangle$ and $[g, {}_n h_i] \in H$ $\forall g \in G$, some integer n, $i = 1, \ldots, m$ then $H \text{ sn } HA$ if

(i) A is finite.

(ii) A is a torsion - free minimax group, and $A \cap H = 1$.

(iii) A is a direct sum of r groups of type C_{p^∞}.

Proof Since $A \cap H \triangleleft G$, we may assume that $A \cap H = 1$. Let $C = C_G(A)$. Then $H \cap C \triangleleft G$ and we may assume $H \cap C = 1$. Then

$$H \cong H/H \cap C \cong HC/C$$

and we may regard H as a subgroup of the automorphism group of A.

Case (i). The automorphism group of A is finite and so H is finite. Hence by theorem 5.1, $H \text{ sn } G$.

Case (ii). $A^p \text{ char } A \triangleleft G \Rightarrow A^p \triangleleft G$.

Now A/A^p is the direct product of cyclic groups (Kaplansky [14] Theorem 6). Hence, since A is a minimax group, A has finite rank and so A/A^p is a finite elementary abelian p - group. By case (i)

$$HA^p/A^p \ \mathrm{sn} \ HA/A^p$$

and so $[A, {}_nH] \leq HA^p$ for some integer n.

Therefore,

$$\begin{aligned}[A, {}_nH] &\leq HA^p \cap A \\ &= A^p(H \cap A) \\ &= A^p\end{aligned}$$

Let A have rank r, then $[A, {}_rH] \leq A^p$.

This is true for all primes p and so

$$[A, {}_rH] \leq \bigcap_p A^p$$

But by Robinson [24] Theorem 4.31 $\bigcap_p A^p = 1$, since A is a torsion - free abelian minimax group.

Therefore $[A, {}_rH] = 1$, and H sn G.

Case (iii). Regarding A additively H may be regarded as a group of linear transformations of an r - dimensional vector space V, over the field of p - adic numbers. (See [25] p. 79.)

Suppose that the action of H on V is irreducible. Then by Mal'cev's structure theorem for soluble linear groups (See [25] theorem 3.21) H is abelian - by - finite. Moreover, with respect to a suitable basis of V, H has a normal subgroup N, of finite index such that N consists of diagonal matrices.

Now an element $g \in GL(r, F)$, F a field, is called unipotent if $(g - 1_r)^r = 0$.

Since $[a, {}_nh_i] \in H \cap A = 1$, in additive notation this means that h_i has unipotent action on A.

So $A(h_i^{\,s} - 1_r)^r = 0$ for every integer $s \geq 0$.

Let $|H : N| = t$. Then $H^t \leq N$ and so $h_i^{\,t}$ can be represented as a diagonal matrix. But, by the above, the action of $h_i^{\,t}$ on V is unipotent and so all its eigenvalues are 1. So $h_i^{\,t}$ can be represented by the unit matrix 1_r.

Hence $h_i^{\,t} = 1$ and this is true for all i: $1 \leq i \leq m$.

Now for each i, $< h_i >$ sn $< h_i >^A$ (by, for example, Peng [20]).

Hence $[\, A, \underbrace{< h_i >, \ldots, < h_i >}_{i} \,] = 1$.

But $< h_i >$ is periodic. Therefore, applying lemma 5.9 we have, since A is radicable, that

$$[\, A, < h_i > \,] = 1 .$$

So $[\, A, H \,] = 1$ and H acts trivially on V.

Now let H act reducibly on V, and form an FH – composition series in V, where F is the field of p – adic numbers. Then, from the above, H acts trivially on each composition factor.

So H can be represented by a group of unitriangular $r \times r$ matrices.

Hence $[\, A, {}_r H \,] = 1$ and so H sn G.

<u>Proof of theorem J</u>

We use induction on d, the derived length of G. If $d = 1$, then G is abelian and so the theorem is trivially true.

So assume $d > 1$ and let N be the last but one term of the derived series of G.

By induction we have

$$HN/N \text{ sn } G/N$$

and so HN sn G.

Since the group HN satisfies the hypotheses of the theorem we may assume $G = HN$. Since $H \cap N \lhd G$ we may factor by $H \cap N$ and assume $H \cap N = 1$.

Let T be the torsion subgroup of N. Then T char $N \lhd G$, and therefore $T \lhd G$. Now N/T is torsion - free, hence by lemma 5.10 (ii) HT/T sn G/T.

Hence we may assume $G = HT$. Let T_p be the p - component of T. T_p char $T \lhd G \Rightarrow T_p \lhd G$, and T_p is a direct product of finitely many cyclic and quasi - cyclic p - groups.

Let F_p be the finite residual of T_p. Then F_p char $T_p \lhd G \Rightarrow F_p \lhd G$.

Since T_p/F_p is finite we have HF_p/F_p sn HT_p/F_p by lemma 5.10(i).

Now F_p is a direct product of finitely many groups of type C_{p^∞}. Hence by lemma 5.10 (iii) H sn HF_p.

Hence H sn HT_p and so for each prime p there exists an integer $n = n(p)$ such that

$$[\, T_p, {}_nH \,] \leq T_p \cap H = 1 .$$

Since $T_p \lhd G$ and $T \in \mathfrak{A}$ we obtain

$$[\, T_p, {}_nH \,] = [\, T_p, {}_nHT \,] = [\, T_p, {}_nG \,] = 1 .$$

So $T_p \leq \zeta_n(G)$.

Now T is an S_1 - group and so has elements of only finitely many distinct prime orders.

Hence we can find an integer s such that

$$T_p \leq \zeta_s(G)$$

for all primes p occurring as orders of elements of T.

Then $T \leq \zeta_s(G)$

and as in previous proofs $H \triangleleft^s HT$, i.e. H sn G.

Appendix

We give a proof of the following result by Brewster [4]. The proof is based on that for the derived series given by Lennox [17], and was communicated to us by Dr. Stewart Stonehewer, whom we thank for his permission to include it here.

Theorem (Brewster) Suppose that G is generated by two subnormal subgroups H and K and

$$G = HK\gamma_r(G) \quad \forall r \geq 1.$$

Then $G = HK$.

We shall use:

Lemma Suppose $G = HK$ with $H \triangleleft^m G$ and $K \triangleleft^n G$. Then, given integers c, d there exists an integer e such that

$$\gamma_e(G) \leq \gamma_c(H)\gamma_d(K).$$

Proof We use induction on m. When $m = 1$, the lemma follows from lemma 2.6 of the thesis.

Hence suppose that $m > 1$.

Now if $H_1 = H^G$, then $H \triangleleft^{m-1} H_1$ and $H_1 = H(H_1 \cap K)$.

Hence by induction, there exists an integer e_1 such that

$$\gamma_{e_1}(H) \leq \gamma_c(H)\gamma_d(H_1 \cap K).$$

But by the case $m = 1$, with H_1 replacing H, we have that there exists an integer e such that

$$\gamma_e(G) \leq \gamma_{e_1}(H)\gamma_d(K).$$

Therefore $\gamma_e(G) \leq \gamma_c(H)\gamma_d(K)$.

Proof of theorem Suppose that $H \triangleleft^m G$. We proceed by induction on m.

Case $m = 1$ Clearly $G = HK$.

Case $m = 2$ Let $K \lhd^n G$. We use induction on n. If $n = 1$, then $G = HK$. Hence we may assume $n > 1$.

Let $H_1 = H^G$, $K_1 = K^G$, $L = H_1 \cap K_1$.

Let $M = H \cap L \lhd H_1$. Then $G = HK_1$ and so $H_1 = H_1 \cap HK_1 = H(H_1 \cap K_1) = HL$.

Let $M^* = M^K$. Since H centralizes L/M, H normalizes M^* and so $M^* \lhd G$.

Hence $H_1 = H^G$ centralizes L/M^* and L/M^* is abelian.

Therefore H_1/HM^* is abelian.

By the lemma, there exists an integer c, such that

$$\gamma_c(G) \leq H_1'K \leq HM^*K.$$

Hence $G = H\gamma_c(G)K \leq HM^*K$ and $G = HM^*K$.

Let $J = \langle M, K \rangle$.

Therefore $G = HJ$ (1)

Since $M^* \lhd G$, $J = M^*K \text{ sn } G$.

Therefore, by the lemma, for any positive integer r, there exists an integer d such that

$$\gamma_d(G) \leq H\gamma_r(J) \quad \text{from (1)}.$$

But by hypothesis $G = H\gamma_d(G)K$ and so $G = H\gamma_r(J)K$.

Therefore, $J = (H \cap J)\gamma_r(J)K \quad \forall r \geq 1$ and

$$J = \langle M, K \rangle \leq \langle H \cap J, K \rangle \leq J.$$

Hence $J = \langle H \cap J, K \rangle$.

Now $J \leq K_1$ and $K \lhd^{n-1} K_1$. Therefore by induction on n

$$J = (H \cap J)K.$$

Then from (1) $G = HK$ as required.

<u>Case $m \geq 3$</u>

Let $H = H_m \lhd H_{m-1} \lhd \cdots \lhd H_1 \lhd G$ be the normal closure series of H in G.

Then, by induction on m, $G = H_{m-1}K$. (2)

Thus, by the lemma, given any integer r there exists an integer s such that

$$\gamma_s(G) \leq \gamma_r(H_{m-1})K.$$

Therefore

$$\begin{aligned} H_i = H\gamma_s(G)K \cap H_1 &\leq H\gamma_r(H_{m-1})K \cap H_1 \\ &\leq H(\gamma_r(H_{m-1})K \cap H_1) \\ &\leq H\gamma_r(H_{m-1})(K \cap H_1) \\ &\leq H\gamma_r(H_1)(K \cap H_1) \leq H_1. \end{aligned}$$

Therefore $H_1 = H\gamma_r(H_1)(K \cap H_1) \quad \forall\, r$ (3)

But by (2) $H_1 = H_{m-1}(K \cap H_1)$ and

$H_{m-2} = H_{m-1}(K \cap H_{m-2})$.

$$\begin{aligned} \text{So } H_i = H^{H_{m-2}(K \cap H_1)} &= H^{K \cap H_{m-2}(K \cap H_i)} \\ &\leq \langle H, K \cap H_1 \rangle \quad \text{since } m \geq 3 \\ &\leq H_1. \end{aligned}$$

Hence $H_1 = \langle H, K \cap H_1 \rangle$ and therefore induction on m and (3) give

$$H_1 = H(K \cap H_1).$$

Therefore $G = H_1 K = H(K \cap H_1)K = HK$

as required.

Bibliography

1. R. Baer — Nilpotent Groups and their generalizations. Trans. Amer. Math. Soc. 47, 393–434 (1940)

2. R. Baer — Nilgruppen. Math.Z. 62 (1955) 402–437

3. R. Baer — Lokal Noethersche Gruppen. Math.Z. 66 (1957), 341-363

4. D. Brewster — A Criterion for the Permutability of Subnormal Subgroups. J. Algebra (To appear)

5. M. Drukker, D.J.S. Robinson, I.N. Stewart — The Subnormal Coalescence of some classes of groups of finite rank. J. Austral. Math.Soc. 16, 324-327 (1973)

6. K.W. Gruenberg — The Engel Elements of a Soluble Group. Illinois J. Math. 3, 151-168 (1959)

7. K.W. Gruenberg — The Upper Central Series in Soluble Groups. Illinois J. Math. 5, 436–466 (1961

8. P. Hall — On Non-Strictly Simple Groups. Proc. Cambridge Philos. Soc. 59, 531-553 (1963)

9. P. Hall, B. Hartley — The Stability Group of a Series of Subgroups. Proc. London Math. Soc. III Ser. 16, 1-39 (1966)

10. B. Hartley, T.A. Peng — Subnormality, Ascendancy and the Minimal Condition. To appear.

11. K.A. Hirsch — On Infinite Soluble Groups IV. J. London Math. Soc. 27, 81–85 (1952)

12. J.A. Hulse — Joins of Ascendant Subgroups. Proc. London Math. Soc. (3) 25, 399–412 (1972)

13. K. Iwasawa — Einige Sätze über freie Gruppen. Proc. Acad. Tokyo. 19 (1943) 272-274

14. I. Kaplansky — Infinite Abelian Groups. Ann Arbor: Univ. Michigan Press 1954

15. O.H. Kegel — Über den Normalisator von Subnormalen und erreichbaren Untergruppen. Math. Ann. 163, 248–258 (1966)

16. A.G. Kurosh — The Theory of Groups, 2nd ed. (2 vols.) New York: Chelsea 1960

17. J.C. Lennox — Ph.D. thesis Cambridge (1970)

18. A.I. Mal'cev — Homomorphisms of Finite Groups. Ivanov. gosudarst. ped. Inst. ucenye. Zapiski. fiz-mat. Nauki 18, 49-60 (1956)

19. H. Neumann — Varieties of Groups. Berlin/Heidelberg/ New York: Springer (1967)

20. T.A. Peng — Subnormality in Soluble-by-Max Groups. To appear.

21. B.I. Plotkin — Radical Groups. Mat. Sb. 37, 507-526 (1955) Amer. Math. Soc. Translations (2) 17, 9-28 (1961)

22. D.J.S. Robinson — Joins of Subnormal Subgroups. Illinois J. Math 9, 144-168 (1965)

23. D.J.S. Robinson — On the Theory of Subnormal Subgroups. Math. Z. 89, 30-51 (1965)

24. D.J.S. Robinson — Residual Properties of some classes of Infinite Soluble Groups. Proc. London Math. Soc. (3) 17, 257-270 (1967)

25. D.J.S. Robinson — Finiteness Conditions and Generalized Soluble Groups, Part 1. Berlin/ Heidelberg/ New York Springer 1972

26. D.J.S. Robinson — Finiteness Conditions and Generalized Soluble Groups, Part 2. Berlin/ Heidelberg/ New York Springer 1972

27. J.E. Roseblade — On Certain Subnormal Coalition Classes. J. Algebra 1, 132-138 (1964)

28. J.E. Roseblade — The Permutability of Orthogonal Subnormal subgroups. Math. Z. 90, 365-372 (1965)

29. J.E. Roseblade — A note on subnormal coalition classes. Math. Z. 90, 373-375 (1965)

30. J.E. Roseblade — The derived series of the join of subnorma subgroups. Math. Z. 117, 57-69 (1970)

31. J.E. Roseblade, S.E. Stonehewer — Subjunctive and Locally Coalescent Classes of groups. J. Algebra 8, 423-435 (1968)

32. S.E. Stonehewer — The Join of Finitely many subnormal subgroups. Bull. London Math. Soc. 2, 77-82 (1970)

33. S.E. Stonehewer — Nilpotent Residuals of Subnormal Subgroups Math. Z. 139, 45-54 (1974)

34. H. Wielandt — Eine Verallgemeinerung der invariante Untergruppen. Math.Z. 45, 209-244 (1939)

35. H. Wielandt — Vertauschbare nachinvariante Untergruppen. Abh. Math. Sem. Univ. Hamburg 21, 55-62 (1957)

36. H. Wielandt — Kriterien für subnormalität in endlichen gruppen. Math.Z. 138, 199-203 (1974)

37. H. Zassenhaus — The Theory of Groups. 2nd ed. New York : Chelsea 1958